Asymptotic localization of energy in non-disordered oscillator chains

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Abstract

We study two popular one-dimensional chains of classical anharmonic oscillators: the rotor chain and a version of the discrete non-linear Schrödinger chain. We assume that the interaction between neighboring oscillators, controlled by the parameter $\epsilon > 0$, is small. We rigorously establish that the thermal conductivity of the chains has a non-perturbative origin, with respect to the coupling constant ϵ, and we provide strong evidence that it decays faster than any power law in ϵ as $\epsilon \to 0$. The weak coupling regime also translates into a high temperature regime, suggesting that the conductivity vanishes faster than any power of the inverse temperature. To our knowledge, it is the first time that a clear connection is established between KAM-like phenomena and thermal conductivity.

1 Introduction

The rigorous derivation of transport properties of solids from molecular dynamics is a big and inspiring challenge in statistical mechanics out of equilibrium. It has been recognized since a long time that the transfer of energy could be strongly reduced, or even suppressed, in some Hamiltonian systems. Anderson localization provides probably the clearest example of this phenomenon. In the context of thermal transport, it is realized in disordered harmonic crystals [5][24], which constitute however a very untypical class of solids, since they are equivalent to an ideal gas of non-interacting phonons. For interacting quantum systems, one expects that the phenomenon of Anderson localization can persist in some regimes, giving rise to the so-called 'many-body localization' [2]. Recently, a mathematical approach to this question was developed in [15]. We learned of this work shortly after starting the present project, and it was a source of inspiration for us, especially for the perturbative part in Section 3. updated reference

At finite volume, Nekhoroshev estimates [26] and the KAM theorem [27] furnish a whole class of classical Hamiltonians that allow energy to be spread through the system only at very slow rates for all initial condition, and not at all for some of them. These results partially extend to finite energy excitations of Hamiltonians depending on infinitely many variables (see [4][11][25] among others). At infinite volume, time-periodic and spatially localized solutions, called breathers, are also known to exist for generic type of classical oscillators chains [22]. In [23], an extensive adiabatic invariant is shown to exist for the Klein Gordon lattice at low but positive temperature (i.e. for initial conditions of infinite energy), implying an asymptotically slow mixing rate of the system. As such however, all these results are of little help to understand the thermal conductivity of solids.

In this paper, we analyze two classical chains of strongly anharmonic oscillators (without disorder), and we show asymptotic localization of energy in a regime characterized by high thermal fluctuations, in comparison with the coupling strength between near atoms. Let $\epsilon > 0$ denote a parameter controlling the strength of the coupling. We establish that energy can only diffuse through these systems at times that are larger than any inverse power of ϵ as $\epsilon \to 0$, except perhaps for a set of states, whose probability is itself smaller than any inverse power in ϵ, with respect to the Gibbs state at a positive temperature T. In that sense, our result could be thought of as an analog of Nekhoroshev estimates, at infinite volume and positive temperature. We hope that these results also provide some complementary view on the slow relaxation to equilibrium observed for chains with strong anharmonic on-site pinning [13].

The first system we consider is a chain of rotors, consisting of particles constrained to move on circles, and weakly coupled through cosine interactions. Numerical studies indicate that this chain behaves as a normal conductor [12], though the conductivity becomes divergent as temperature is sent to zero. The defocusing discrete non-linear Schrödinger chain is the second system we look at. We study this chain in the regime where the on-site anharmonic pining dominates the weak harmonic coupling. Here as well, simulations show this chain to be a normal conductor [16]. It is known that, besides energy, these chains preserve a second quantity (see Section 2 below). To stress that our results do not depend on this, we allow for an extra interaction, that breaks the second conservation law.

Our results ultimately rest on a phenomenon that is at the heart of all the results above: close individual atoms typically oscillate each at different frequencies, so that resonances, that are responsible for energy transfer in a perturbative regime, are rarely observed. To explain this a bit further, we find it useful to introduce a comparison with weakly coupled disordered chains. Assuming there the on-site potential to be harmonic, uncoupled atoms simply oscillate at a fixed but random eigenfrequency. When a small interaction is turned on, only a few disconnected resonant spots are created here and there, corresponding to places where the eigenfrequencies of near atoms are, in very good approximation, in specific ratios with respect to each others. This observation has allowed to conclude to asymptotic localization of energy for a wide class of interaction potentials [14], and, for harmonic interactions, to a true localization [10] as long as the coupling is not too large.

Let us now move back to our non-disordered chains. Since the on-site interaction is strongly anharmonic, each uncoupled atom oscillates at a frequency that depends on its energy. Moreover, in the absence of interaction, the Gibbs state is a product measure, so that the eigenfrequency of each oscillator is here as well chosen randomly. So far, the comparison with inhomogeneous chains is thus perfect. When the interaction is turned on, it is still so that rare resonant spots will appear here and there. However, these resonant islands are no longer attached to a fixed place. Instead, as a bit of energy get transferred, the eigenfrequencies are slightly modified, so that resonant sites can be destroyed here and recreated there. This phenomenon a priori favors the transport of energy. In fact, our main difficulty compared to [14], was to show that this process itself occurs so slowly that it is irrelevant at the time scales we consider. Once this difficulty is overcome, the results of this paper resemble very closely the analogous statements in [14]. Those results were in turn inspired by [21] where the weak coupling limit for oscillator chains with energy-conserving dynamics was analyzed rigorously for the first time.

The paper is organized as follows. Our results are stated in Section 2. The rest of the paper is devoted to the proofs. We have not been able to handle both chains in a unified way, though it is mainly a question of details. As a consequence, Sections 3 to 5 exclusively deal with the rotor chain, while the non-linear Schrödinger chain is considered in Section 6.

In Section 3, a KAM-like change of variables is constructed, that isolates from the rest the part of the interaction giving rise to resonances. The stability of resonant islands is studied in Section 4. Our main result is finally shown in Section 5 for the rotor chain. Adaptations needed to handle the non-linear Schrödinger chain are explained in Section 6. The final Section 7 contains the proof of three corollaries.

2 Models and Results

We define precisely the chains under study as well as the thermal conductivity, and state our results together with some comments.

2.1 Models

Let $N \geq 1$ be an odd integer and let $\mathbb{Z}_N = \{-(N-1)/2, \ldots, (N-1)/2\}$. Let also $\gamma \geq 0$. For $\gamma = 0$, the two dynamics we study preserve both the total energy and a second quantity: the momentum for the rotor chain, and the ℓ^2-norm for the discrete non-linear Schrödinger chain. When $\gamma > 0$, these extra conservation laws are broken, so that only energy remains conserved. We will assume free boundary conditions, though that is only a matter of convenience: all our conclusions would still hold for other choices of boundary conditions.

The rotor chain. The phase space consists of the points

$$(q,\omega) \ = \ (q_x,\omega_x)_{x\in\mathbb{Z}_N} \ \in \ \Omega = \Omega_N = (\mathbb{T}\times\mathbb{R})^N \quad \text{with} \quad \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}).$$

The Hamiltonian is

$$\begin{aligned} H(q,\omega) \ &= \ D(\omega) + \epsilon V(q,\omega) \ = \ \sum_{x\in\mathbb{Z}_N} H_x(q,\omega) \ = \ \sum_{x\in\mathbb{Z}_N} \big(D_x(\omega) + \epsilon V_x(q)\big) \\ &= \ \frac{1}{2}\sum_{x\in\mathbb{Z}_N} \omega_x^2 \ + \ \epsilon \sum_{x\in\mathbb{Z}_N} \Big(\gamma(1-\cos q_x) + \big(1-\cos(q_x - q_{x+1})\big)\Big), \end{aligned} \tag{1}$$

with the convention $q_{\pm(N+1)/2} = q_{\pm(N-1)/2}$ (free boundary conditions on both sides). The Hamilton equations of motion are

$$\dot{q} \ = \ \nabla_\omega H \qquad \text{and} \qquad \dot{\omega} \ = \ -\nabla_q H. \tag{2}$$

The total momentum $\sum_x \omega_x$ is a conserved quantity only at $\gamma = 0$. Given an initial condition $(q,\omega)\in\Omega$, we denote the Hamiltonian flow by $(X_\epsilon^t(q,\omega))_{t\geq 0} \subset \Omega$.

The discrete non-linear Schrödinger chain. The phase space consists of the points

$$\psi \ = \ (\psi_x)_{x\in\mathbb{Z}_N} \ \in \ \Omega = \Omega_N = \mathbb{C}^N \simeq (\mathbb{R}^2)^N.$$

The Hamiltonian is

$$\begin{aligned} H(\psi) \ &= \ D(\psi) + \epsilon V(\psi) \ = \ \sum_{x\in\mathbb{Z}_N} H_x(\psi) \ = \ \sum_{x\in\mathbb{Z}_N} \big(D_x(\psi) + \epsilon V_x(\psi)\big) \\ &= \ \frac{1}{2}\sum_{x\in\mathbb{Z}_N} |\psi_x|^4 \ + \ \epsilon \sum_{x\in\mathbb{Z}_N} \Big(\gamma(\psi_x + \overline{\psi}_x)^2 + |\psi_x - \psi_{x+1}|^2\Big), \end{aligned} \tag{3}$$

with again the convention $\psi_{\pm(N+1)/2} = \psi_{\pm(N-1)/2}$. Writing $H(\psi)$ as $H(\psi,\overline{\psi})$, the Hamilton equations of motion take the redundant form

$$i\dot{\psi} \ = \ \nabla_{\overline{\psi}} H \qquad \text{and} \qquad i\dot{\overline{\psi}} \ = \ -\nabla_\psi H.$$

The total ℓ^2-norm $\sum_x |\psi_x|^2$ is a conserved quantity only at $\gamma = 0$. Given an initial condition $\psi\in\Omega$, we denote the Hamiltonian flow by $(X_\epsilon^t(\psi))_{t\geq 0} \subset \Omega$.

To see the analogy between this chain and the rotor chain, we could move to action-angle, or polar, coordinates. Writing

$$\omega_x \ = \ |\psi_x|^2 \qquad \text{and} \qquad \tan q_x \ = \ \Im\psi_x/\Re\psi_x,$$

the Hamiltonian (3) is is recast as

$$H(q,\omega) \;=\; \sum_{x\in\mathbb{Z}_N}\Big(\frac{\omega_x^2}{2}+4\gamma\epsilon\omega_x\cos^2 q_x+\epsilon\big(\omega_x+\omega_{x+1}-2\sqrt{\omega_x\omega_{x+1}}\cos(q_x-q_{x+1})\big)\Big)$$

while Hamilton equations now precisely take the form (2). Unfortunately, this change of variable is not well defined if some frequency ω_x vanishes, implying that the field $\nabla_\omega H$ becomes singular as $\omega_x\to 0$. We will, for this reason, not make explicitly use of it.

2.2 Heat current and thermal conductivity

Given two functions $f,g\in\mathcal{C}^\infty(\Omega)$, we define, for rotors,

$$L_f g \;=\; \{f,g\} \;=\; \nabla_\omega f\cdot\nabla_q g-\nabla_q f\cdot\nabla_\omega g \;=\; -\{g,f\} \;=\; -L_g f, \tag{4}$$

and for the non-linear Schrödinger chain,

$$L_f g \;=\; \{f,g\} \;=\; -i\big(\nabla_{\overline{\psi}} f\cdot\nabla_\psi g-\nabla_\psi f\cdot\nabla_{\overline{\psi}} g\big). \tag{5}$$

Given $a\in\mathbb{Z}_N$, we define the energy current $\epsilon J_{a,a+1}$ across the bond $(a,a+1)$ by

$$\epsilon J_{a,a+1} \;=\; L_H\sum_{x>a}H_x \;=\; \Big\{\sum_{y\leq a}H_y\,,\,\sum_{x>a}H_x\Big\} \;=\; \{H_a,H_{a+1}\}. \tag{6}$$

We then define the total, normalized, current $\epsilon\mathcal{J}$ by

$$\epsilon\mathcal{J} \;=\; \frac{\epsilon}{N^{1/2}}\sum_{a\in\mathbb{Z}_N}J_{a,a+1}.$$

Let $T>0$ be some fixed temperature. The Gibbs state is a measure on Ω defined, for the rotor chain, by

$$f \;\mapsto\; \langle f\rangle_T \;=\; \frac{1}{Z(T)}\int_\Omega f(q,\omega)\,\mathrm{e}^{-H(q,\omega)/T}\,\mathrm{d}q\mathrm{d}\omega,$$

where $Z(T)$ is a normalization factor such that this measure is a probability measure. For the non-linear Schrödinger chain, the expression is analogous: $H(q,\omega)$ is replaced by $H(\psi)$, and $\mathrm{d}q\mathrm{d}\omega$ is replaced by $\mathrm{d}\Re(\psi)\mathrm{d}\Im(\psi)$. The Green-Kubo conductivity of the system is defined, if the limits exist, as a space-time variance [20]:

$$\kappa(T,\epsilon) \;=\; \lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{T^2}\Big\langle\Big(\frac{\epsilon}{\sqrt{t}}\int_0^t \mathcal{J}_N(X_\epsilon^s)\,\mathrm{d}s\Big)^2\Big\rangle_T \tag{7}$$

where we have written $\mathcal{J}_N$ instead of $\mathcal{J}$, to remind ourselves that this quantity depends on N, and we have set $\mathcal{J}_N(X_\epsilon^s)=\mathcal{J}_N\circ X_\epsilon^s$. We note that, thanks to good decorrelation properties of the Gibbs measure (see Section 7), the limit $N\to\infty$ is independent of the boundary conditions.

2.3 Results

We start by an abstract result expressing that, in all orders in perturbation in ϵ, only local oscillations of the energy field (and hence no persistent currents) can be produced by the dynamics.

Theorem 1. *Let the Hamiltonian be given by* (1) *or* (3)*. Let $T > 0$ be fixed. Choose any $n \geq 1$ and let then $\mathrm{C}_n < +\infty$ be large enough. For any $N \geq 1$ and $a \in \mathbb{Z}_N$, the current across the bond $(a, a+1)$ can be decomposed as*

$$\epsilon J_{a,a+1} \;=\; L_H U_a + \epsilon^{n+1} G_a$$

The functions U_a and G_a are smooth, of zero average, $\langle U_a \rangle_T = \langle G_a \rangle_T = 0$, and they depend only on variables labeled by $z \in \mathbb{Z}_N$ with $|z - a| \leq \mathrm{C}_n$, and satisfy the bounds

$$\langle U_a^2 \rangle_T \;\leq\; \mathrm{C}_n \epsilon^{1/4}, \qquad \langle (\partial_\sharp U_a)^2 \rangle_T \;\leq\; \mathrm{C}_n \epsilon^{-1/4}, \qquad \langle G_a^2 \rangle_T \;\leq\; \mathrm{C}_n, \qquad \langle (\partial_\sharp G_a)^2 \rangle_T \;\leq\; \mathrm{C}_n \tag{8}$$

where $\sharp$ stands for any of the variables.

We deduce two results on the thermal conductivity from this abstract statement. The analysis of the conductivity as defined by (7) is probably out of reach at the present time. We can however obtain some conclusion by assuming that the true value of the integral in (7) is already attained at a time t that grows as some inverse power in ϵ as $\epsilon \to 0$. One can argue (see e.g. Chapter 5 of [6]) that this is equivalent to exciting the system locally, and observing the relaxation for a time t of this order. Our result is quite similar in spirit to results about weak coupling limits in such systems, e.g. [17][9][21], where one describes the dynamics in a scaling limit where coupling vanishes but time goes to infinity. However, in our case, these scaling limits are trivial in the sense that we do not see any transport on the time scales that we study. We would find it very interesting to push the analysis to longer time scales and to exhibit a non-vanishing contribution to the conductivity.

So first, we follow the dynamics for a time of order ϵ^{-n}, for an arbitrary large n, and let $\epsilon \to 0$. We believe the next theorem to be a strong indication that $\kappa(T, \epsilon) = \mathcal{O}(\epsilon^m)$ for any $m \geq 1$. To establish this rigorously, one would need to exchange the limits $t \to \infty$ and $\epsilon \to 0$.

Theorem 2. *Let the Hamiltonian be given by* (1) *or* (3)*. Let $T > 0$ be fixed. Let $1 \leq m < n$. Then*

$$\lim_{t\to\infty} \limsup_{\epsilon\to 0} \limsup_{N\to\infty} \frac{\epsilon^{-m}}{T^2} \Big\langle \Big(\frac{\epsilon}{\sqrt{\epsilon^{-n} t}} \int_0^{\epsilon^{-n} t} \mathcal{J}_N(X_\epsilon^s)\, \mathrm{d}s \Big)^2 \Big\rangle_T \;=\; 0.$$

One could speculate whether some non-perturbative effects could lead to a breakdown of the conjecture $\kappa(T, \epsilon) = \mathcal{O}(\epsilon^m)$. We cannot exclude this, and in fact we do not even rigorously know whether the chains we consider are normal conductors for some $\epsilon > 0$, that is, whether $\kappa(T, \epsilon) < \infty$. It is however commonly believed that, on sufficiently large time scales, the dynamics of such systems becomes chaotic. As in [14], we can, for the rotor chain, mimic this hypothetic non-perturbative chaotic behavior by a stochastic noise that conserves energy, and that becomes perceptible on very large time scales, namely $\epsilon^{-(n+1)}$, for some arbitrarily large n. We are then able to show that the conductivity is finite and not larger than ϵ^n, so that it can be attributed to the noise. Instead, we do not know what could be the effect of non-perturbative integrable structures, such as solitons traveling ballistically.

Let us consider the rotor chain. For $n \geq 1$, we let

$$\mathcal{L} \;=\; L_H \;+\; \epsilon^{n+1} S \qquad \text{with} \qquad S u(q,\omega) \;=\; \sum_{x\in\mathbb{Z}_N} \big(u(q, \dots, -\omega_x, \dots) - u(q,\omega) \big) \tag{9}$$

be the generator of a Markov process on Ω. Let us denote by $(\mathcal{X}^t_\epsilon(q,\omega))_{t\geq 0}$ the Markov process generated by $\mathcal{L}$ and started from the point (q,ω). We denote by E the expectation with respect to the realizations of the noise S.

Theorem 3. *Let the Hamiltonian be given by* (1)*. Let $T>0$ be fixed. For any $n\geq 1$, it holds that there is $\mathrm{C}_n<\infty$ such that, for sufficiently small $\epsilon>0$,*

$$\lim_{t\to\infty}\limsup_{N\to\infty}\frac{1}{T^2}\Big\langle \mathsf{E}\Big(\frac{\epsilon}{\sqrt{t}}\int_0^t \mathcal{J}_N(\mathcal{X}^s_\epsilon)\,\mathrm{d}s\Big)^2\Big\rangle_T \;\leq\; \mathrm{C}_n\epsilon^n$$

From Theorem 1, we can also deduce a statement that mirrors the well-known Nekhorohsev theorem for systems consisting of a finite number of degrees of freedom (see e.g. [26]). We recall that the latter states that, for all initial conditions, the action coordinates of the uncoupled system remain $\mathrm{C}\epsilon^b$-close to their original value for a time $\mathrm{e}^{\mathrm{C}(1/\epsilon)^a}$, for some $a,b>0$ and with ϵ the coupling strength. We can reproduce this statement for arbitrary polynomial times, rather than exponential ones, and for a set of configurations that has large probability with respect to the Gibbs state. In [8], a similar result was obtained, limited however to much shorter time scales. Let $I=\{a_1,a_1+1,\dots,a_2\}\subset\mathbb{Z}_N$ be a discrete interval, and let $H_I=\sum_x H_x$. Then

Theorem 4. *Let the Hamiltonian be given by* (1) *or* (3)*. Let $T>0$ be fixed. For any $n\geq 1$, there is $\mathrm{C}_n<\infty$ such that, for sufficiently small $\epsilon>0$, and for any I as above,* very slightly reformulated

$$\Big\langle\big(H_I(X^t_\epsilon)-H_I\big)^2\Big\rangle_T\leq \mathrm{C}_n\epsilon^{1/4},\qquad \textit{for any}\quad 0\leq t\leq\epsilon^{-n}. \tag{10}$$

2.4 Remarks

Temperature dependence. The behavior of the thermal conductivity $\kappa(T,\epsilon)$ defined by (7) as $\epsilon\to 0$ for fixed $T>0$, is directly connected to its behavior as $T\to\infty$ for fixed $\epsilon>0$. Indeed, assuming that (7) is well defined, we have as we will see that, for every $\sigma>0$,

$$\kappa(\epsilon,T) \;=\; \frac{1}{\sigma}\,\kappa(\sigma^2\epsilon,\sigma^2T)\qquad\text{for the rotor chain,} \tag{11}$$

$$\kappa(\epsilon,T) \;=\; \frac{1}{\sigma}\,\kappa(\sigma\epsilon,\sigma^2T)\qquad\text{for the non-linear Schrödinger chain.} \tag{12}$$

We therefore also conjecture for the two chains that $\kappa(T,\epsilon\sim 1)=\mathcal{O}(1/T^m)$ for every $m\geq 1$ as $T\to\infty$. As one can check from the calculations below, we obtain also scaling relations like (11), (12) for the finite-time approximations to the conductivity κ that figure in Theorem 2, so we could literally restate this result for the high-temperature regime. An analogous scaling result was obtained in [1] for a different chain.

Let us see how to obtain (11). Let $\sigma>0$. Let us write H_ϵ instead of H to explicitly keep track of the coupling strength. It is computed that, if $(q(t),\omega(t))_{t\geq 0}$ is a solution to Hamilton's equation (2) for the Hamiltonian H_ϵ given by (1), then $(q'(t),\omega'(t))_{t\geq 0}$ given by

$$q'(t)\;=\;q(\sigma t),\qquad \omega'(t)\;=\;\sigma\,\omega(\sigma t),$$

solves Hamilton's equations for the Hamiltonian $H_{\sigma^2\epsilon}$. It is then computed by means of (6) that $\epsilon J_{a,a+1} = \epsilon\omega_{a+1}\sin(q_a - q_{a+1})$, where $\epsilon J_{a,a+1}$ denotes the current through $(a, a+1)$ corresponding to the Hamiltonian H_ϵ. Let us denote by $\sigma^2\epsilon J'_{a,a+1}$ the current corresponding to $H_{\sigma^2\epsilon}(q',\omega')$. It holds that

$$\frac{\epsilon}{\sqrt{t}}\int_0^t \mathcal{J}(q(s),\omega(s))\,\mathrm{d}s \;=\; \frac{1}{\sigma^{5/2}}\cdot\frac{\sigma^2\epsilon}{\sqrt{t'}}\int_0^{t'}\mathcal{J}'(q'(s'),\omega'(s'))\,\mathrm{d}s' \qquad \text{with} \qquad t' = t/\sigma.$$

In the Gibbs measure, the change of variables implies the change $T \mapsto \sigma^2 T$ for the temperature:

$$\frac{\int u(q,\omega)\mathrm{e}^{-H_\epsilon(q,\omega)/T}\,\mathrm{d}q\mathrm{d}\omega}{\int \mathrm{e}^{-H_\epsilon(q,\omega)/T}\,\mathrm{d}q\mathrm{d}\omega} \;=\; \frac{\int u(q',\omega')\mathrm{e}^{-H_{\sigma^2\epsilon}(q',\omega')/\sigma^2 T}\,\mathrm{d}q'\mathrm{d}\omega'}{\int \mathrm{e}^{-H_{\sigma^2\epsilon}(q',\omega')/\sigma^2 T}\,\mathrm{d}q'\mathrm{d}\omega'}.$$

The scaling relation (11) then follows from the definition (7). The scaling relation (12) is obtained analogously: it is here observed that, if $(\psi(t))_{t\geq 0}$ is a solution to Hamilton's equations for the hamiltonian H_ϵ given by (3), then $(\psi'(t) = \sqrt{\sigma}\psi(\sigma t))_{t\geq 0}$ solves Hamilton's equations for the Hamiltonian $H_{\sigma\epsilon}$.

Higher dimensions. We conjecture that our results extend to higher dimensional lattices. The arguments in Sections 3 and 4 would indeed carry over straightforwardly. The evolution of energy appears thus equally frozen for two or three-dimensional lattices as for a one-dimensional one. Unfortunately, the proof of Theorem 1 that appears in Section 5, does not extend as such to higher dimensions. Although the problem seems to us purely technical and we find it very plausible that one can adapt it to higher dimensions, we have not pursued this here.

Other models. ***Rem. by W:*** **Stylistic changes in this remark** ***EOR*** Our results depend mainly on three properties of the models: First, the dynamics of isolated oscillators is one-dimensional, and thus integrable, so that the frequency of oscillation is a well defined concept. Second, the isolated isolators are strongly anharmonic, implying that the frequencies depend on the energy in a non-trivial way. Third, the coupling is weak, so that perturbation theory applies. It is thus natural to ask whether, for example, our results would also hold for the Hamiltonian

$$H(q,p) \;=\; \sum_{x\in\mathbb{Z}_N}\Big(\frac{p_x^2}{2} + \frac{q_x^4}{4} + \frac{\epsilon}{2}(q_x - q_{x+1})^2\Big), \tag{13}$$

as it possesses the three listed properties, i.e. whether its conductivity is also non-perturbative as $\epsilon \to 0$.

It turns out that we actually exploit a specific characteristic of the chains that we look at: the perturbation only involves a finite number of combinations of the eigenfrequencies of neighboring oscillators, meaning technically that we may work with finite trigonometric polynomials (see Section 3). This would not longer be true for the chain defined by (13), for which trigonometric polynomials should be replaced by more generic analytic functions. While this extra difficulty can be overcome in usual KAM or Nekhoroshev theorems, part of our proof would likely break down (see Section 4). The generalisation of our theorems to the chain defined by (13) appears thus to us as an open question.

How optimal are our bounds ? It is numerically observed that the chains under study are normal conductors [12][16], so that we expect localization of energy to be at best asymptotic. Still, the time scales at which energy starts diffusing could be much larger than any inverse power in ϵ. At finite volume for example, Nekhoroshev estimates imply the absence of diffusion over exponentially long times. However, in [3], the thermal conductivity of a classical non-linear disordered chain is studied, and it is argued that the localization is broken at a scale that is roughly of the order of $\mathrm{e}^{-c\ln^3(1/\epsilon)}$. Since we expect the energy to travel more easily in the non-disordered chains thanks to the mobility of resonants spots, we conjecture that, here as well, localization does not persist on longer times than that. In other words, we do not think that one can obtain Nekhoroshev estimates in infinite volume for times as long as those in finite volume.

3 Approximate change of variables

We introduce an auxiliary Hamiltonian $\widetilde{H} = \widetilde{H}_{n_1}$, defined for an arbitrary $n_1 \geq 1$, and give the needed links between $\widetilde{H}$ and the original Hamiltonian H. We first introduce some definitions, then state the results, and finally prove them. The formulas introduced in the second part are probably best demystified by first reading the beginning of the proof. It is seen there that we define a KAM-like formal change of variable. In contrast to the KAM-scheme however, our expansion is only perturbative, and does not involve any renormalization of the energy of individual atoms at each step.

3.1 Preliminary definitions

Throughout all this work, we will deal with functions f in a subspace $\mathcal{S}(\Omega)$ of $\mathcal{C}^\infty(\Omega)$. A function f belongs to $\mathcal{S}(\Omega)$ if the three following conditions are realized for some number

$$r = r(f) > 0. \tag{14}$$

1. The function f is a sum of local terms:

$$f \;=\; \sum_{x\in\mathbb{Z}_N} f_x \qquad \text{with} \qquad \frac{\partial f_x}{\partial q_y} \;=\; \frac{\partial f_x}{\partial \omega_y} \;=\; 0 \qquad \text{if} \qquad |x-y| > r(f). \tag{15}$$

   This decomposition is not unique. Obviously, this property will be helpful only when $r(f) \ll N$. ***Rem. by W:*** **Just to avoid the suspicion that we have missed somehting here *EOR***

2. The function f depends on the variable q through a finite number of Fourier modes only:

$$f(q,\omega) \;=\; \sum_{\mathrm{k}\in\mathbb{Z}^N} \widehat{f}(\mathrm{k},\omega)\mathrm{e}^{i\mathrm{k}\cdot q} \qquad \text{with} \qquad \widehat{f}(\mathrm{k},\omega) \;=\; 0 \;\; \text{if} \;\; \max_x |\mathrm{k}_x| \;\geq\; r(f). \tag{16}$$

   As a consequence of the spatial locality in 1., it also holds $\widehat{f}(\mathrm{k},\omega) = 0$ as soon as $\mathrm{supp}(\mathrm{k})$ cannot be included in a ball of radius r, where $\mathrm{supp}(\mathrm{k}) = \{x \in \mathbb{Z}_N : \mathrm{k}_x \neq 0\}$.

3. Given any $m \geq 1$, and given any differential operator D, with either $\mathrm{D} = \mathrm{Id}$ or $\mathrm{D} = \partial_{\sharp_1}\dots\partial_{\sharp_m}$, where $\sharp_k$ stands for any of the variables, there is a polynomial p_{D} on $\mathbb{R}^{2r+1}$ so that, for every $x \in \mathbb{Z}_N$, and $(q,\omega) \in \Omega$, Made more precise

$$|\mathrm{D} f_x(q,\omega)| \;\leq\; |p_{\mathrm{D}}(\omega_{x-r},\dots,\omega_{x+r})|. \tag{17}$$

Let $\rho \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ be a smooth cut-off function: $\rho(-x) = \rho(x)$ for every $x \in \mathbb{R}$, $\rho(x) = 1$ for every $x \in [-1,1]$ and $\rho(x) = 0$ for every $x \notin [-2,2]$. For any $a > 0$, we define also ρ_a by $\rho_a(x) = \rho(x/a)$.

Let $0 < \delta < 1$. In this section, we assume this number to be independent of ϵ. We define an operator $\mathcal{R}$ on $\mathcal{S}(\Omega)$ that acts as

$$(\mathcal{R}f)(q,\omega) \;=\; \sum_{\mathrm{k}\in\mathbb{Z}^N} \rho_\delta(\mathrm{k}\cdot\omega)\widehat{f}(\mathrm{k},\omega)\mathrm{e}^{i\mathrm{k}\cdot q}.$$

Let D be the function defined in (1). Given $f \in \mathcal{S}(\Omega)$, the equation

$$L_D u \;=\; (\mathrm{Id} - \mathcal{R})f,$$

where $L_D = \{D, \cdot\}$ is defined in (4), can be solved in $\mathcal{S}(\Omega)$. A solution u is given by

$$u(q,\omega) \;=\; \sum_{\mathrm{k}\in\mathbb{Z}^N} \frac{1-\rho_\delta(\mathrm{k}\cdot\omega)}{i\,\mathrm{k}\cdot\omega}\widehat{f}(\mathrm{k},\omega)\mathrm{e}^{i\mathrm{k}\cdot q},$$

where the sum only goes over terms for which $\mathrm{k}\cdot\omega \neq 0$. This is the only solution such that $\widehat{u}(0,\omega) = 0$ for all $\omega \in \mathbb{R}^N$; we will refer to it as the solution to the equation $L_D u = (\mathrm{Id} - \mathcal{R})f$.

Finally, we will find it convenient to work with formal power series in ϵ: given a vector space E, these are expressions of the form $Y = \sum_{k\ge 0} \epsilon^k Y^{(k)}$, where $Y^{(k)} \in E$ for every $k \ge 0$. We naturally extend algebraic operations in E to operations between formal series. Given $l \ge 0$ and given a formal series Y, we define the truncation

$$\mathcal{T}_l(Y) \;=\; \sum_{k=0}^{l} \epsilon^k Y^{(k)} \;\in\; E. \tag{18}$$

If a formal power series Y is such that $Y^{(k)} = 0$ for all $k > l$ for some $l \in \mathbb{N}$, we will allow ourselves to identify Y with its truncation $\mathcal{T}_l(Y) \in E$.

3.2 Statement of the results

Given $k \ge 1$, let $\pi(k) \subset \mathbb{N}^k$ be the collection of k-tupels $\underline{j} = (j_l)_{l=1,\dots,k}$ of nonnegative integers satisfying the constraint

$$\sum_{l=1}^{k} l j_l = k,$$

in particular $0 \le j_l \le k$.

For $k \ge 0$, we recursively define operators $Q^{(k)}, R^{(k)}$ and $S^{(k)}$ on $\mathcal{S}(\Omega)$, as well as functions $U^{(k)} \in \mathcal{S}(\Omega)$. Here and below, let us adopt the convention $A^0 = \mathrm{Id}$ for an operator A. We first set $Q^{(0)} = R^{(0)} = \mathrm{Id}$, $S^{(0)} = 0$ and $U^{(0)} = 0$. Next, for $k \ge 1$, we define $U^{(k)}$ as the solution to the equation

$$L_D U^{(k)} \;=\; (\mathrm{Id} - \mathcal{R})\big(S^{(k-1)}D + Q^{(k-1)}V\big). \tag{19}$$

and then set

$$Q^{(k)} = \sum_{\underline{j}\in\pi(k)} \frac{1}{j_1!\dots j_k!} L_{U^{(k)}}^{j_k} \dots L_{U^{(1)}}^{j_1}, \tag{20}$$

$$R^{(k)} = \sum_{\underline{j}\in\pi(k)} \frac{(-1)^{j_1+\dots+j_k}}{j_1!\dots j_k!} L_{U^{(1)}}^{j_1} \dots L_{U^{(k)}}^{j_k}, \tag{21}$$

$$S^{(k)} = \sum_{\substack{\underline{j}\in\pi(k+1):\\ j_{k+1}=0}} \frac{1}{j_1!\dots j_k!} L_{U^{(k)}}^{j_k} \dots L_{U^{(1)}}^{j_1}. \tag{22}$$

For $n_1 \geq 1$, we define

$$\widetilde{H} = \widetilde{H}_{n_1} = D + \sum_{k=1}^{n_1} \epsilon^k \mathcal{R}(S^{(k-1)}D + Q^{(k-1)}V). \tag{23}$$

The following Proposition is shown in Subsection 3.3 below.

Proposition 1. *Let us consider the formal series $R = \sum_{k\geq 0} \epsilon^k R^{(k)}$ of operators on $\mathcal{S}(\Omega)$.*

1. $H = \mathcal{T}_{n_1}\big(R\widetilde{H}_{n_1}\big)$.
2. *For every $f = \sum_{k=0}^{n_1} \epsilon^k f^{(k)}$, it holds that*

$$L_H\big(\mathcal{T}_{n_1}(Rf)\big) = \mathcal{T}_{n_1}\big(RL_{\widetilde{H}_{n_1}} f\big) + \epsilon^{n_1+1} L_V \sum_{k=0}^{n_1} R^{(n_1-k)} f^{(k)}.$$

3. *The function $\widetilde{H}$ is symmetric under the exchange $\omega \mapsto -\omega$. Moreover, for any $k \geq 0$, the operator $R^{(k)}$ maps symmetric functions with respect to this operation, to symmetric functions.*

The function $\widetilde{H}$ and the formal operator R have several characteristic that are good to remember.

1. Both $\widetilde{H}$ and R are expressed as power series in ϵ, as is seen from (23) and from the definition of R given in Proposition 1. We introduce also the notation

$$\widetilde{H} = \sum_{k=0}^{n_1} \epsilon^k \widetilde{H}^{(k)} \qquad \text{with} \qquad \widetilde{H}^{(0)} = D, \qquad \widetilde{H}^{(k)} = \mathcal{R}(S^{(k-1)}D + Q^{(k-1)}V) \quad \text{for} \quad k \geq 1.$$

2. For each $k \geq 0$, $\widetilde{H}^{(k)}$ is a function belonging to $\mathcal{S}(\Omega)$, and $R^{(k)}$ an operator on $\mathcal{S}(\Omega)$. Let $f = \sum_{x\in\mathbb{Z}_N} f_x \in \mathcal{S}(\Omega)$ be given. The functions $\widetilde{H}^{(k)}$ and $R^{(k)}f$ can be decomposed as a sum of local terms, with for example, for $k \geq 1$,

$$\widetilde{H}_x^{(k)} = \mathcal{R}(S^{(k-1)}D_x + Q^{(k-1)}V_x) \qquad \text{and} \qquad (R^{(k)}f)_x = R^{(k)}f_x.$$

   Moreover, we will show in Subsection 3.3 below, that there exists an integer r_k such that

$$r\big(\widetilde{H}^{(k)}\big) \leq r_k \qquad \text{and} \qquad r\big(R^{(k)}f\big) \leq r_k + r(f), \tag{24}$$

   where r is the parameter introduced in (14).

3. The function $\widetilde{H}^{(k)}$ and the operator $R^{(k)}$ depend on δ if $k \geq 1$. Let thus $k \geq 1$. In what follows, we will use the symbol b to denote smooth, bounded functions on $\mathbb{R}^N \times (0,1)$ with bounded derivatives of all order ***Rem. by W:*** **reformulation to avoid the impression that there is a uniform bound** ***EOR*** , and the symbol f for functions in $\mathcal{S}(\Omega)$. We will show the two following assertions in Subsection 3.3 below. First, there is an integer m_k such that, given $x \in \mathbb{Z}_N$, $\widetilde{H}^{(k)}_x$ can be expressed as a sum of the type

$$\widetilde{H}^{(k)}_x(q,\omega;\delta) \;=\; \delta^{-2(k-1)} \sum_{j=1}^{m_k} b_{j,x}(\omega/\delta,\delta)\, f_{j,x}(q,\omega). \tag{25}$$

such that the functions $b_{j,x}$ and $f_{j,x}$ depend on the same variables and the same Fourier modes as $\widetilde{H}^{(k)}_x$ and the bounds on them can be chosen uniformly in x. Second, consider a function $g(\,\cdot\,;\delta) \in \mathcal{S}(\Omega)$ such that $g_x(q,\omega;\delta) = b_x(\omega/\delta,\delta) f_x(q,\omega)$. Then, there is an integer $m_{k,g}$ such that $(R^{(k)}g)_x$ can be expressed as a sum of the type

$$(R^{(k)}g)_x(q,\omega;\delta) \;=\; \delta^{-2k} \sum_{j=1}^{m_{k,g}} \tilde{b}_{j,x}(\omega/\delta,\delta) \tilde{f}_{j,x}(q,\omega). \tag{26}$$

and such that the functions $\tilde{b}_{j,x}$ and $\tilde{f}_{j,x}$ depend on the same variables and the same Fourier modes as $(R^{(k)}g)_x$. ***Rem. by W:*** **I changed the apostrophes into tildes because later on I used the apostrophes for derivatives** ***EOR***

3.3 Proof of Proposition 1 and relations (24-26)

Proof of Proposition 1. Given a function $U \in \mathcal{S}(\Omega)$, a formal change of variable, seen as an operator on $\mathcal{S}(\Omega)$, is defined through

$$\mathrm{e}^{\epsilon L_U} \;=\; \sum_{k\geq 0} \frac{\epsilon^k}{k!} L_U^k.$$

Given now a sequence $(U^{(k)})_{k\geq 1} \subset \mathcal{S}(\Omega)$, that we later will identify with the sequence defined by (19), we construct the formal change of variable

$$\begin{aligned} Q \;&=\; \dots \mathrm{e}^{\epsilon^n L_{U^{(n)}}} \dots \mathrm{e}^{\epsilon^2 L_{U^{(2)}}} \mathrm{e}^{\epsilon L_{U^{(1)}}} \;=\; \sum_{j_1\geq 0,\dots,j_n\geq 0,\dots} \frac{\epsilon^{j_1+\dots+nj_n+\dots}}{j_1!\dots j_n!\dots} \big(\dots L^{j_n}_{U^{(n)}} \dots L^{j_1}_{U^{(1)}}\big) \\ &=\; \mathrm{Id} + \sum_{k\geq 1} \epsilon^k \sum_{\underline{j}\in\pi(k)} \frac{1}{j_1!\dots j_k!} L^{j_k}_{U^{(k)}} \dots L^{j_1}_{U^{(1)}} \;=\; \sum_{k\geq 0} \epsilon^k Q^{(k)}. \end{aligned}$$

The second line was obtained using the definition of $\underline{j} \in \pi(k)$ introduced at the beginning of Subsection 3.2. The formal inverse of Q is given by

$$\begin{aligned} R \;&=\; \mathrm{e}^{-\epsilon L_{U^{(1)}}} \mathrm{e}^{-\epsilon^2 L_{U^{(2)}}} \dots \mathrm{e}^{-\epsilon^n L_{U^{(n)}}} \dots \\ &=\; \mathrm{Id} + \sum_{k\geq 1} \epsilon^k \sum_{\underline{j}\in\pi(k)} \frac{(-1)^{j_1+\dots+j_n}}{j_1!\dots j_k!} L^{j_1}_{U^{(1)}} \dots L^{j_k}_{U^{(k)}} \;=\; \sum_{k\geq 0} \epsilon^k R^{(k)}. \end{aligned}$$

Let us show the first part of Proposition 1. The operators Q and R are formal inverse of each others, so that, for every $f \in \mathcal{S}(\Omega)$ such that $\mathcal{T}_{n_1} f = f$ (see the remark after (18)), it holds that

$$f \;=\; \mathcal{T}_{n_1}\big(R\mathcal{T}_{n_1}(Qf)\big),$$

as can be checked by a direct computations with formal series. We will thus be done if we show that

$$\widetilde{H}_{n_1} \;=\; \mathcal{T}_{n_1}(QH) \tag{27}$$

We compute

$$QH \;=\; \sum_{k\geq 0}\epsilon^k Q^{(k)}(D+\epsilon V) \;=\; D+\sum_{k\geq 1}\epsilon^k (Q^{(k)}D+Q^{(k-1)}V).$$

It holds that

$$Q^{(k)} \;=\; S^{(k-1)}+L_{U^{(k)}} \quad \text{for} \quad k\geq 1.$$

Since $L_{U^{(k)}}D=-L_D U^{(k)}$ for every $k\geq 1$, and taking now $U^{(k)}$ as defined by (19), we obtain

$$QH \;=\; D+\sum_{k\geq 1}\epsilon^k (S^{(k-1)}D+Q^{(k-1)}V-L_D U^{(k)}) \;=\; D+\sum_{k\geq 1}\epsilon^k \mathcal{R}(S^{(k-1)}D+Q^{(k-1)}V).$$

From this, we derive (27).

Let us then show the second part of Proposition 1. The operators Q and R are formal canonical transformations, inverse of each other. Therefore

$$L_H R \;=\; R L_{QH}, \tag{28}$$

as a direct, but lengthy, computation with formal series can confirm. Let us next take f such that $f=\mathcal{T}_{n_1}(f)$. By (27), we find that

$$\mathcal{T}_{n_1}\big(RL_{\widetilde{H}_{n_1}}f\big) \;=\; \mathcal{T}_{n_1}\big(RL_{\mathcal{T}_{n_1}QH}f\big) \;=\; \mathcal{T}_{n_1}\big(RL_{QH}f\big),$$

since higher order terms do not contributre thanks to the overall truncation $\mathcal{T}_{n_1}$. Therefore, by (28),

$$L_H\big(\mathcal{T}_{n_1}(Rf)\big)-\mathcal{T}_{n_1}\big(RL_{\widetilde{H}_{n_1}}f\big) \;=\; L_H\big(\mathcal{T}_{n_1}(Rf)\big)-\mathcal{T}_{n_1}\big(RL_{QH}f\big) \;=\; L_H\big(\mathcal{T}_{n_1}(Rf)\big)-\mathcal{T}_{n_1}\big(L_H Rf\big).$$

Since $L_H=L_D+\epsilon L_V$, it is finally computed that

$$L_H\big(\mathcal{T}_{n_1}(Rf)\big)-\mathcal{T}_{n_1}\big(L_H Rf\big) \;=\; \epsilon^{n_1+1}L_V\sum_{k=0}^{n_1}R^{n_1-k}f^k.$$

Let us finally establish the last part of Proposition 1. A function will be said symmetric or antisymmetric if it is symmetric or antisymmetric with respect to the operation $\omega\mapsto -\omega$. We observe that, if a function U is symmetric, then L_U exchanges symmetric and antisymmetric functions, while L_U preserves the symmetry if U is antisymmetric. The action of $\mathcal{R}$ also preserves the symmetry. We deduce that the operation $L_D^{-1}(\mathrm{Id}-\mathcal{R})$ exchange symmetric and antisymmetric functions, since D is symmetric. It is then recursively established from (19-22) that the functions U_k are antisymmetric for $k\geq 0$, while the operators Q_k, R_k and S_k preserve the symmetry for $k\geq 0$. Since D and V are symmetric, we conclude from (23) that $\widetilde{H}$ is symmetric. $\square$

Proof of (24-26). Let us first establish (24). Given two functions $f,g\in\mathcal{S}(\Omega)$, the function $L_g f$ is decomposed as a sum of local terms $(L_g f)_x$, that we have chosen to be given by $(L_g f)_x=L_g f_x$. A direct computations shows that $r(L_g f)\leq 2r(g)+r(f)$. Since $r(L_D^{-1}(\mathrm{Id}-\mathcal{R})f)=r(f)$, we readily deduce (24) from (19-22).

Let us next show (25) and (26). Since we are only interested in tracking the dependence on δ, we may simplify notations as much as possible in the following way. We use the symbols b and f with the same meaning as in the paragraph where (25) and (26) are stated. Let $n \geq 0$. First, if $g \in \mathcal{S}(\Omega)$, we just write $g \sim \delta^{-n}$ to express that g is of the following form: $g = \sum_x g_x$ as in (15) and g_x take the form $g_x(q,\omega;\delta) = \delta^{-n} \sum_j b_{j,x}(\omega/\delta,\delta) f_{j,x}(\omega,q)$ with all bounds on $b_{j,x}, f_{j,x}$ uniform in x. Next, if A is an operator on $\mathcal{S}(\Omega)$, we just write $A \sim \delta^{-n}$ to express that, for any $h \in \mathcal{S}(\Omega)$ such that $h \sim \delta^{-m}$, we have $Ah \sim \delta^{-n-m}$.

We now observe that, if $g \sim \delta^{-n}$ and $h \sim \delta^{-m}$, then $L_g h \sim \delta^{-(n+m+1)}$, and that if u solves the equation $L_D u = (\mathrm{Id} - \mathcal{R})g$ and if $g \sim \delta^{-n}$, then $u \sim \delta^{-(n+1)}$. It is then established recursively that, for $k \geq 1$, we have

$$Q^{(k-1)} \sim \delta^{-2(k-1)}, \qquad R^{(k-1)} \sim \delta^{-2(k-1)}, \qquad S^{(k-1)}D \sim \delta^{-2(k-1)}, \qquad U^{(k)} \sim \delta^{-(2k-1)}, \tag{29}$$

from which (25) and (26) are readily derived. By the definitions, the relations (29) hold for $k=1$. Let us see that the claim for $1,\dots,k \geq 1$ implies the claim for $k+1$.

Let us start with $Q^{(k)}$. For $\underline{j} \in \pi(k)$, we get from the definition (20) that

$$Q^{(k)} \sim (\delta^{-(2k-1)-1})^{j_k} \dots (\delta^{-(2-1)-1})^{j_1} = \delta^{-2k}.$$

The case of $R^{(k)}$ is handled in the same way. Let us then treat $S^{(k)}D$. We decompose $S^{(k)}D = \sum_{\underline{j}} S^{(k)}_{\underline{j}} D$ according to the definition (22), we pick one of the sequences $\underline{j}$, and we let $l \geq 1$ be the smallest integer such that $j_l \geq 1$, such that the constraint on $\underline{j}$ is $j_1 + 2j_2 + \dots + kj_k = k+1$. Thanks to (19) and to our inductive hypothesis, we get, for some constant $\mathrm{C}(\underline{j})$,

$$\begin{aligned} S^{(k)}_{\pi} D &= \mathrm{C}(\underline{j})\, L^{j_k}_{U^{(k)}} \dots L^{j_l-1}_{U^{(l)}} \big(L_{U^{(l)}} D\big) = -\mathrm{C}(\underline{j})\, L^{j_k}_{U^{(k)}} \dots L^{j_l-1}_{U^{(l)}} \big((\mathrm{Id} - \mathcal{R})(S^{(l-1)})D + Q^{(l-1)}V\big) \\ &\sim \delta^{-2\{l(j_l-1)+\dots+kj_k\}} \delta^{-2(l-1)} = \delta^{-2(k+1)+2l-2(l-1)} = \delta^{-2k}. \end{aligned}$$

So we conclude that $S^{(k)}D \sim \delta^{-2k}$. The statement for $U^{(k+1)}$ is finally derived using (19). □

4 Resonant frequencies

Given a point $x \in \mathbb{Z}_N$, we construct a subset $\mathsf{R}(x)$ of the frequencies ω, seen as a subset of the full phase space Ω that does not depend on the positions q, with the two following characteristics. First, if a state does not belong to this set, then the energy current for the Hamiltonian $\widetilde{H}$ vanishes through the bonds near x. Second, it is approximately invariant under the dynamics generated by $\widetilde{H}$, meaning that in a small time interval, only the frequencies in a subset $\mathsf{S}(x)$, of small probability with respect to the Gibbs measure, can leave or enter the set $\mathsf{R}(x)$.

In our opinion, the ideas of this Section are best understood visually. We hope that figure 1 will help in that respect (see below for the definition of the set $\mathsf{B}(\mathrm{k}_1,\mathrm{k}_2)$). We let

$$r = r(n_1) = \max_{1 \leq k \leq n_1} r_k, \tag{30}$$

where the numbers r_k are defined in (24). We let $\delta > 0$ be as in Section 3.

4.1 Preliminary definitions

We recall that, given $\mathrm{k} \in \mathbb{Z}^N$, we denote by $\mathrm{supp}(\mathrm{k}) \subset \mathbb{Z}^N$ the set of points x such that $\mathrm{k}_x \neq 0$. We define the set $K_r \subset \mathbb{Z}^N$ of vectors $\mathrm{k} = (\mathrm{k}_x)_{x \in \mathbb{Z}_N}$ such that $\max_{x \in \mathbb{Z}_N} |\mathrm{k}_x| \leq r$ and $\mathrm{supp}(\mathrm{k}) \subset \mathrm{B}(r)$ for some ball $\mathrm{B}(r)$ of radius r. We write $|\mathrm{k}|_2^2 = \sum_x |\mathrm{k}_x|^2$. One easily checks that for any $\mathrm{k} \in K_r$ and $r > 1$, we have $|\mathrm{k}|_2 \leq r^2$ and this will be used without further comment.

Given $x \in \mathbb{Z}^d$, we say that a subset $\{\mathrm{k}_1, \dots, \mathrm{k}_p\} \subset K_r$ is a cluster around x if

1. the vectors $\mathrm{k}_1, \dots, \mathrm{k}_p$ are linearly independent, definition
2. if $p \geq 2$, for all $1 \leq i \neq j \leq p$, there exist $1 \leq i_1, \dots, i_m \leq p$ such that $i_1 = i$, $i_m = j$ and $\mathrm{supp}(\mathrm{k}_{i_s}) \cap \mathrm{supp}(\mathrm{k}_{i_{s+1}}) \neq \varnothing$ for all $1 \leq s \leq m-1$, made independent of
3. $\mathrm{supp}(\mathrm{k}_j) \subset \mathrm{B}(x, 4r)$ for some $1 \leq j \leq p$. numeration

Finally, given $\mathrm{k} \in K_r$, we define

$$\pi(\mathrm{k}) \;=\; \{\omega \in \mathbb{R}^N : \mathrm{k} \cdot \omega = 0\}.$$

Given a subspace $E \subset \mathbb{R}^N$, and given $\omega \in \mathbb{R}^N$, we denote by $P(\omega, E)$ the orthogonal projection of ω on the subspace E.

4.2 Approximately invariant sets of resonant frequencies

Let $L > 0$, let $n_2 \geq 1$, and let $x \in \mathbb{Z}_N$. Let us define two subsets of $\mathbb{R}^N$: a set $\mathsf{R}_{\delta,n_2}(x) \subset \mathbb{R}^N$ of resonant frequencies, and a small set $\mathsf{S}_{\delta,n_2}(x) \subset \mathbb{R}^N$ of "multi-resonant" frequencies.

To define $\mathsf{R}_{\delta,n_2}(x)$, let us first define the sets $\mathsf{B}_\delta(\mathrm{k}_1, \dots, \mathrm{k}_p) \subset \mathbb{R}^N$, where $\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$ is a cluster around x. We say that $\omega \in \mathsf{B}_\delta(\mathrm{k}_1, \dots, \mathrm{k}_p)$ if

$$\big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \dots \cap \pi(\mathrm{k}_p)\big)\big|_2 \;\leq\; L^p \delta \tag{31}$$

and if, for every linearly independent $\mathrm{k}'_1, \dots, \mathrm{k}'_{p'} \in K_r \cap \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$,

$$\big|P\big(\omega, \pi(\mathrm{k}'_1) \cap \dots \cap \pi(\mathrm{k}'_{p'})\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \dots \cap \pi(\mathrm{k}_p)\big)\big|_2 \;\leq\; \big(L^p - L^{p'}\big)\delta.$$

We next define $\mathsf{R}_{\delta,n_2}(x)$ as the union of all the sets $\mathsf{B}_\delta(\mathrm{k}_1, \dots, \mathrm{k}_p) \subset \mathbb{R}^N$ with $p \leq n_2$.

We then define $\mathsf{S}_{\delta,n_2}(x)$ as the set of points $\omega \in \mathbb{R}^N$ for which there exists a cluster $\{\mathrm{k}_1, \dots, \mathrm{k}_{n_2}\}$ around x, such that $|\mathrm{k}_j \cdot \omega| \leq L^{n_2+1}\delta$ for every $1 \leq j \leq n_2$.

We finally define a smooth indicator function of the complement of $\mathsf{R}_{\delta,n_2}(x)$ by means of a convolution:

$$\theta_{x,\delta,n_2}(\omega) \;=\; 1 - \frac{1}{\Big(\int_{\mathbb{R}} \rho_\delta(z)\,\mathrm{d}z\Big)^N} \int_{\mathbb{R}^N} \chi_{\mathsf{R}_{\delta,n_2}(x)}(\omega + \omega') \Big(\prod_{x \in \mathbb{Z}_N} \rho_\delta(\omega'_x)\Big)\,\mathrm{d}\omega'. \tag{32}$$

This naturally may be seen as a function on the full phase space Ω that is independent of the $q-$variable.

Proposition 2. *Let n_1 be given, and so $r(n_1)$ defined by (30) be fixed as well. Let then $n_2 \geq 1$ be fixed. The following holds for L large enough.*

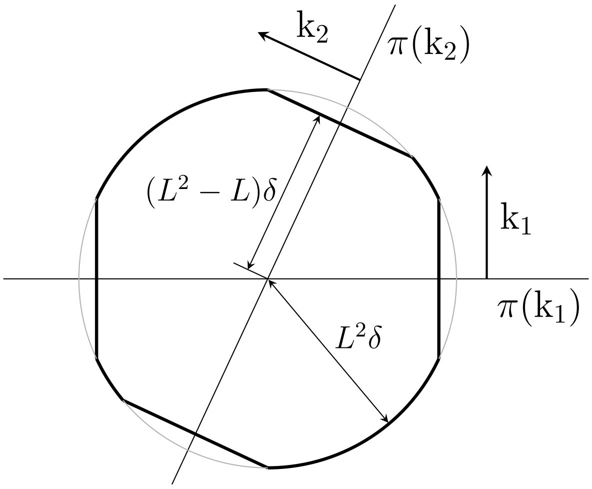

Figure 1: The set $\mathsf{B}_\delta(k_1, k_2)$. The plane is the subspace of points of the form $\omega - P(\omega, \pi(k_1) \cap \pi(k_2))$, for $\omega \in \mathbb{R}^N$. We have drawn a disk of radius $L^2\delta$ that is 'flattened' by an amount $L\delta$ at the intersection of its boundary with the lines $\pi(k_1), \pi(k_2)$. To simplify the figure, we have pretended that k_1 and k_2 are the only vectors in $\mathrm{span}\{k_1, k_2\} \cap K_r$. This is not so in reality: the disk still needs to be flattened by an amount $L\delta$ at each intersection point of its boundary with a line $\pi(k)$ for all $k \in \mathrm{span}\{k_1, k_2\} \cap K_r$. The set $\mathsf{B}(k_1, k_2, k_3)$ could be similarly visualized as a ball of radius $L^3\delta$, that is flattened by $L\delta$ along the circles corresponding to the intersection of its boundary with a plane $\pi(k)$, and flattened by $L^2\delta$ at the points where its boundary intersect a line $\pi(k) \cap \pi(k')$, for all $k, k' \in \mathrm{span}\{k_1, k_2, k_3\} \cap K_r$.

1. *If* $\theta_{x,\delta,n_2}(\omega) > 0$ *then* $\rho_\delta(\omega \cdot k) = 0$ *for all* $k \in K_r$ *such that* $\mathrm{supp}(k) \subset \mathrm{B}(x, 4r)$.

2. $L_{\widetilde{H}_{n_1}} \theta_{x,\delta,n_2}(q, \omega) = 0$ *for all* $(q, \omega) \in \Omega$ *such that* $q \in \mathbb{T}^N$ *and* $\omega \notin \mathsf{S}_{n_2}(x)$.

4.3 Proof of Proposition 2

We start by a series of lemmas. The first one simply expresses, in a particular case, that if a point is close to two vector spaces, then it is also close to their intersection. The uniformity of the constant C comes from the fact that we impose the vectors to sit in the set K_r.

Lemma 1. *Let* $p \geq 1$. *There exists a constant* $\mathrm{C} = \mathrm{C}(r, p) < +\infty$ *such that, given linearly independent vectors* $k_1, \dots, k_p, k_{p+1} \in K_r$ *and given* $\omega \in \mathbb{R}^N$, *it holds that*

$$\big|\omega - P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_p) \cap \pi(k_{p+1})\big)\big|_2 \;\leq\; \mathrm{C}\,\Big(|k_{p+1} \cdot \omega| + \big|\omega - P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_p)\big)\big|_2\Big).$$

Proof. First,

$$\begin{aligned}\big|\omega - P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_{p+1})\big)\big|_2 \;\leq\;& \big|\omega - P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_p)\big)\big|_2 \\ &+ \big|P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_{p+1})\big) - P\big(\omega, \pi(k_1) \cap \dots \cap \pi(k_p)\big)\big|_2. \quad (33)\end{aligned}$$

The lemma is already shown if the second term in the right hand side is zero. We further assume this not to be the case. Next, since $\mathrm{k}_{p+1} \cdot P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big) = 0$, we obtain

$$\begin{aligned} \mathrm{k}_{p+1} \cdot \omega \;=\; & \mathrm{k}_{p+1} \cdot \Big(\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\Big) + \\ & \qquad \mathrm{k}_{p+1} \cdot \Big(P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\Big). \end{aligned}$$

This implies

$$\begin{aligned} \Big|\mathrm{k}_{p+1} \cdot \Big(P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\Big)\Big| \;\leq \\ |\mathrm{k}_{p+1} \cdot \omega| + |\mathrm{k}_{p+1}|_2 \Big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\Big|_2. \end{aligned} \tag{34}$$

The vector

$$v \;=\; \frac{P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)}{\big|P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2} \tag{35}$$

is well defined since we have assumed that the denominator in this expression does not vanish. The bound (34) is rewritten as

$$\begin{aligned} \big|P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \;\leq \\ \frac{|\mathrm{k}_{p+1} \cdot \omega| + |\mathrm{k}_{p+1}|_2 \Big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\Big|_2}{|\mathrm{k}_{p+1} \cdot v|} \end{aligned}$$

Inserting this last inequality in (33), we arrive at

$$\big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \;\leq\; \frac{|\mathrm{k}_{p+1} \cdot \omega|}{|\mathrm{k}_{p+1} \cdot v|} + \left(1 + \frac{|\mathrm{k}_{p+1}|_2}{|\mathrm{k}_{p+1} \cdot v|}\right)\Big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\Big|_2.$$

To finish the proof, it remains to establish that $|\mathrm{k}_{p+1} \cdot v|$ can be bounded from below by some strictly positive constant, where v is given by (35). Let us show that

$$v = \pm \frac{P\big(\mathrm{k}_{p+1}, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)}{\big|P\big(\mathrm{k}_{p+1}, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\big|_2}. \tag{36}$$

We can find vectors $\mathrm{k}_{p+2}, \ldots, \mathrm{k}_N$ so that $\{\mathrm{k}_1, \ldots, \mathrm{k}_N\}$ forms a basis of $\mathbb{R}^N$ and so that every vector k_j with $p+2 \leq j \leq N$ is orthrogonal to $\mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_{p+1}\}$. We express the vector ω in this basis, $\omega = \sum_{j=1}^N \omega^j \mathrm{k}_j$, and, from (35), we deduce that, for some non-zero constant R, we have

$$v \;=\; R \sum_{j=1}^{N} \omega^j \Big\{P\big(\mathrm{k}_j, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big) - P\big(\mathrm{k}_j, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\Big\}.$$

All the terms corresponding to $1 \leq j \leq p$ vanish since $\mathrm{k}_j \perp \pi(\mathrm{k}_j)$, the term $P\big(\mathrm{k}_{p+1}, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)$ vanishes for the same reason, and all the terms corresponding to $j \geq p+2$ vanish too, as they read in fact $\mathrm{k}_j - \mathrm{k}_j = 0$. So, the only term left is $R\omega^{p+1} P\big(\mathrm{k}_{p+1}, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)$ and, since $|v| = 1$, we arrive at (36).

From (36) we deduce that

$$|v \cdot \mathrm{k}_{p+1}| \;=\; \big|P\big(\mathrm{k}_{p+1}, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)\big)\big|_2.$$

If $\mathrm{k}_{p+1} \perp \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$, then the right hand side just becomes $|\mathrm{k}_{p+1}|_2$. This quantity is bounded from below by a strictly positive constant since so is the the norm of any nonzero vector in K_r. Otherwise, if $\mathrm{k}_{p+1} \not\perp \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$, we know however that the quantity cannot vanish since $\mathrm{k}_{p+1} \notin \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$. Because they are only finitely many vectors $\mathrm{k} \in K_r$ with the property that $\mathrm{k} \not\perp \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$, we conclude that the quantity is bounded from below by a strictly positive constant. □

The next Lemma describes the crucial geometrical properties of the sets $\mathsf{B}_\delta(\mathrm{k}_1, \ldots, \mathrm{k}_p)$ that allows to establish the second assertion of Proposition 2.

Lemma 2. *Let $\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$ be a cluster around x. If, given $\mathrm{K} < +\infty$, L is taken large enough, then, for every $\mathrm{k} \in K_r \cap \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$, it holds that*

$$\omega \in \mathsf{B}_\delta(\mathrm{k}_1, \ldots, \mathrm{k}_p) \quad \textit{and} \quad |\mathrm{k} \cdot \omega| \;\le\; \mathrm{K}\delta \qquad \Rightarrow \qquad \omega + t\mathrm{k} \in \mathsf{B}_\delta(\mathrm{k}_1, \ldots, \mathrm{k}_p) \quad \textit{as long as} \quad |t| \le \delta.$$

Proof of Lemma 2. To simplify some further expressions, let us define

$$\omega' = \omega - P(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_p)) \in \mathrm{span}(\mathrm{k}_1, \ldots, \mathrm{k}_p).$$

The conditions ensuring that $\omega \in \mathsf{B}(\mathrm{k}_1, \ldots, \mathrm{k}_p)$ now simply read

$$|\omega'|_2 \;\le\; L^p\delta \qquad \text{and} \qquad \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big)\big|_2 \;\le\; \big(L^p - L^{p'}\big)\delta \qquad (p' < p), \tag{37}$$

for all linearly independent $\mathrm{k}'_1, \ldots, \mathrm{k}'_{p'} \in \mathrm{k} \in K_r \cap \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$. The condition $|\mathrm{k} \cdot \omega| \le \mathrm{K}\delta$ implies $|\mathrm{k} \cdot \omega'| \le \mathrm{K}\delta$. We need to show that precisions added

$$|\omega' + t\mathrm{k}|_2 \;\le\; L^p\delta \qquad \text{for} \qquad |t| \le \delta, \tag{38}$$

$$\big|P\big(\omega' + t\mathrm{k}, \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big)\big|_2 \;\le\; \big(L^p - L^{p'}\big)\delta \qquad \text{for} \qquad |t| \le \delta. \tag{39}$$

Let us start with (38): expicit proof of (38) added

$$\begin{aligned} |\omega' + t\mathrm{k}|_2 \;&\le\; |\omega' - P(\omega', \pi(\mathrm{k}))|_2 + |P(\omega', \pi(\mathrm{k}))|_2 + |t||k|_2 \;\le\; |\omega' \cdot \mathrm{k}| + |P(\omega', \pi(\mathrm{k}))|_2 + |t||k|_2 \\ &\le\; \mathrm{K}\delta + (L^p - L)\delta + r^2\delta \;\le\; L^p\delta. \end{aligned}$$

Here, to get the penultimate inequality, we have used (37) and the hypothesis $\mathrm{k} \in K_r \cap \mathrm{span}\{\mathrm{k}_1, \ldots, \mathrm{k}_p\}$, implying in particular $|\mathrm{k}|_2 \le r^2$, while the last inequality is valid for large enough L.

Let us next move to (39). Let us fix $\mathrm{k}'_1, \ldots, \mathrm{k}'_{p'}$. It is seen that, if $\mathrm{k} \in \mathrm{span}\{\mathrm{k}'_1, \ldots, \mathrm{k}'_{p'}\}$, then (39) is actually satisfied for all $t \in \mathbb{R}$. Let us therefore assume $\mathrm{k} \notin \mathrm{span}\{\mathrm{k}'_1, \ldots, \mathrm{k}'_{p'}\}$. We write also $\mathrm{k} = \mathrm{k}_{p'+1}$. We will show that, because $|\mathrm{k} \cdot \omega| \;\le\; \mathrm{K}\delta$, then in fact slightly reformulated

$$\big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big)\big|_2 \;\le\; \big(L^p - L^{p'} - r^2\big)\delta. \tag{40}$$

Since $|\mathrm{k}|_2 \le r^2$, this will imply (39).

To establish (40), we start by writing the decompositions

$$|\omega'|_2^2 \;= \big|\omega' - P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big)\big|_2^2 + \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big)\big|_2^2, \tag{41}$$

$$|\omega'|_2^2 \;= \big|\omega' - P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'+1})\big)\big|_2^2 + \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'+1})\big)\big|_2^2. \tag{42}$$

We bound the first term in the right hand side of (42) by applying Lemma 1 and then using (41): misprint corrected

$$\begin{aligned}\big|\omega' - P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p+1})\big)\big|_2^2 \;&\le\; \mathrm{C}\Big(|\mathrm{k}\cdot\omega'|^2 + \big|\omega' - P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_p)\big)\big|_2^2\Big) \\ &\le\; \mathrm{C}\Big(|\mathrm{k}\cdot\omega'|^2 + |\omega'|_2^2 - \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_p)\big)\big|_2^2\Big).\end{aligned}$$

It may be assumed that $\mathrm{C} \ge 1$. Reinserting this bound in (42) yields

$$\begin{aligned}|\omega'|_2^2 \;&\le\; \mathrm{C}\Big(|\mathrm{k}\cdot\omega'|^2 + |\omega'|_2^2 - \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_p)\big)\big|_2^2\Big) + \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'+1})\big)\big|_2^2 \\ &\le\; \mathrm{C}\Big(\mathrm{K}^2\delta^2 + |\omega'|_2^2 - \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_p)\big)\big|_2^2\Big) + (L^p - L^{p'+1})^2\delta^2. \qquad (43)\end{aligned}$$

where the hypotheses $|\mathrm{k}\cdot\omega'| \le \mathrm{K}\delta$ and $\omega \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$ have been used to get the last line.

Let us now show that (43) implies (40) for L large enough. For this let us write

$$\begin{aligned}|\omega'| \;&=\; (1-\mu)^{1/2} L^p\delta \qquad \text{with} \qquad 0 \le \mu \le 1, \\ \big|P\big(\omega', \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_p)\big)\big|_2 \;&=\; (1-\nu)^{1/2}(L^p - L^{p'} - r^2)\delta \qquad \text{with} \qquad \nu \le 1\end{aligned}$$

($\mu > 0$ actually, thanks to the hypothesis $|\mathrm{k}\cdot\omega'| \le \mathrm{K}\delta$, see figure 1). Showing (40) amounts showing $\nu \ge 0$. With these new notations, inequality (43) is rewritten as

$$\begin{aligned}1 + (\mathrm{C}-1)\mu \;\le 1 &- \frac{2}{L^{p-p'-1}} \\ &+ \frac{1}{L^{2(p-p'-1)}} + \mathrm{C}\bigg(\frac{\mathrm{K}^2}{L^{2p}} + \frac{2}{L^{p-p'}} + \frac{2r^2}{L^p} - \frac{1}{L^{2(p-p')}} - \frac{r^4}{L^{2p}} - \frac{2r^2}{L^{2p-p'}}\bigg) \\ &+ \mathrm{C}\nu\bigg(1 - \frac{L^{p'} + r^2}{L^p}\bigg)^2.\end{aligned}$$

The left hand side is larger or equal to 1. But, when L becomes large, the right hand side is larger or equal to 1 only if $\nu > 0$. □

Lemma 3. *Let* $\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$ *be a cluster around* x*, and let* $\mathrm{k} \in K_r$ *be such that* $\mathrm{k} \notin \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$*, but such that* $\{\mathrm{k}_1, \dots, \mathrm{k}_p, \mathrm{k}\}$ *is a cluster. If, given* $\mathrm{K} < +\infty$*,* L *is taken large enough, then*

$$\omega \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p) \quad \textit{and} \quad |\mathrm{k}\cdot\omega| \;\le\; \mathrm{K}\delta \qquad \Rightarrow \qquad \omega \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p, \mathrm{k}).$$

Proof of Lemma 3. Let us write $\mathrm{k} = \mathrm{k}_{p+1}$. Let $\omega \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$. By Lemma 1 and by hypothesis, it holds that

$$\big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \;\le\; \mathrm{C}\,(\mathrm{K}\delta + L^p\delta) \;\le\; (L-1)L^p\delta$$

if L is large enough. Then

$$\big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \;\le\; L^{p+1}\delta$$

and, for every $\mathrm{k}'_1, \dots, \mathrm{k}'_{p'} \in K_r \cap \mathrm{span}(\mathrm{k}_1, \dots, \mathrm{k}_{p+1})$, with $p' < p+1$,

$$\begin{aligned}\big|P\big(\omega, \pi(\mathrm{k}'_1) \cap \cdots \cap \pi(\mathrm{k}'_{p'})\big) - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \;&\le\; \big|\omega - P\big(\omega, \pi(\mathrm{k}_1) \cap \cdots \cap \pi(\mathrm{k}_{p+1})\big)\big|_2 \\ &\le\; (L^{p+1} - L^p)\delta \;\le\; (L^{p+1} - L^{p'})\delta.\end{aligned}$$

This shows $\omega \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p, \mathrm{k})$. $\square$

Proof of Proposition 2. Let us start with the first claim. Let $\mathrm{k} \in K_r$ be such that $\mathrm{supp}(\mathrm{k}) \subset \mathrm{B}(x, 4r)$, and let $\omega \in \mathbb{R}^N$ be such that $\theta_x(\omega) > 0$. On the one hand, from the definition (32) of θ_x, it holds that there exists $\omega' \in \mathbb{R}^N$, with $\max_x |\omega'_x| \le 2\delta$, such that $\omega + \omega' \notin \mathsf{R}(x)$. On another hand, since $\mathrm{supp}(\mathrm{k}) \subset \mathrm{B}(x, 4r)$, we conclude that $\{\mathrm{k}\}$ alone is a cluster around x so that, if $\omega'' \in \mathbb{R}^N$ is such that

$$\frac{|\mathrm{k} \cdot \omega''|}{|\mathrm{k}|_2} \;=\; \big|\omega'' - P(\omega'', \pi(\mathrm{k}))\big| \;\le\; L\delta,$$

then $\omega'' \in \mathsf{R}(x)$. We thus conclude that $|(\omega + \omega') \cdot \mathrm{k}| > |\mathrm{k}|_2 L\delta \ge L\delta$, and so that

$$|\omega \cdot \mathrm{k}| \;=\; |(\omega + \omega') \cdot \mathrm{k} \;-\; \omega' \cdot \mathrm{k}| \;\ge\; L\delta - 2\delta r^2 \;>\; 2\delta$$

if L is large enough. We conclude that $\rho_\delta(\omega \cdot \mathrm{k}) = 0$.

Let us then show the second part of the Proposition. Since, by (23) and (24), the Hamiltonian $\widetilde{H}$ takes the form

$$\widetilde{H}(q, \omega) \;=\; \sum_{\mathrm{k} \in K_r} \rho_\delta(\mathrm{k} \cdot \omega) G(\mathrm{k}, \omega) \mathrm{e}^{i\mathrm{k} \cdot q},$$

for some function G on $\mathbb{Z}^N \times \mathbb{R}^N$, and since the function θ_x is independent of the $q-$variable, it holds that

$$L_{\widetilde{H}_{n_1}} \theta_x(q, \omega) \;=\; -\nabla_q \widetilde{H} \cdot \nabla_\omega \theta_x(q, \omega) \;=\; -i \sum_{\mathrm{k} \in K_r} \big(\mathrm{k} \cdot \nabla_\omega \theta_x(\omega)\big) \rho_\delta(\mathrm{k} \cdot \omega) G(\mathrm{k}, \omega) \mathrm{e}^{i\mathrm{k} \cdot q}.$$

It is thus enough to show that

$$\mathrm{k} \cdot \nabla_\omega \theta_x(\omega) = 0 \quad \text{for every} \quad \mathrm{k} \in K_r \quad \text{and every} \quad \omega \notin \mathsf{S}_{n_2}(x) \quad \text{such that} \quad |\mathrm{k} \cdot \omega| \le 2\delta.$$

Let us thus fix $\omega \in \mathbb{R}^N$ and $\mathrm{k} \in K_r$ with these restrictions. By definition (32), we see that $\mathrm{k} \cdot \nabla_\omega \theta_x(\omega) = 0$ if, for every $\omega' \in \mathbb{R}^N$ such that $\max_x |\omega'_x| \le 4\delta$, it holds that

$$\omega + \omega' \in \mathsf{R}_{n_2}(x) \qquad \Rightarrow \qquad \omega + \omega' + t\mathrm{k} \in \mathsf{R}_{n_2}(x) \quad \text{for all } t \text{ such that } |t| \text{ is small enough.}$$

Here, the maximal value allowed for $|t|$ may depend on ω but not on ω'. We distinguish three cases: either at least one of the cases 1 and 2 is realized, or, if none of them is realized, than case 3 is.

1. There exists a cluster $\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$ around x, with $p \le n_2$, such that $\omega + \omega' \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$ and that $\mathrm{k} \perp \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$. It is then seen from the definition of $\mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$ that, for every $t \in \mathbb{R}$, $\omega + \omega' + t\mathrm{k} \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$. Therefore $\omega + \omega' + t\mathrm{k} \in \mathsf{R}_{n_2}(x)$ for every $t \in \mathbb{R}$.

2. There exists a cluster $\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$ around x, with $p \le n_2$, such that $\omega + \omega' \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$ and that $\mathrm{k} \in \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$. Since $|\mathrm{k} \cdot \omega| \le 2\delta$ and since $\max_x |\omega_x| \le 4\delta$, it holds that $|\mathrm{k} \cdot (\omega + \omega')| \le (4r^2 + 2)\delta$. Then, by Lemma 2, for $|t| \le \delta$ we still have $\omega + \omega' + t\mathrm{k} \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$ if L was chosen large enough. Therefore $\omega + \omega' + t\mathrm{k} \in \mathsf{R}_{n_2}(x)$ for $|t| \le \delta$.

3. For any cluster $\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$ around x, with $p \le n_2$, such that $\omega + \omega' \in \mathsf{B}(\mathrm{k}_1, \dots, \mathrm{k}_p)$, it holds that $\mathrm{k} \notin \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$, and that $\mathrm{k} \not\perp \mathrm{span}\{\mathrm{k}_1, \dots, \mathrm{k}_p\}$. Let us see that, since we assume that $\omega \notin \mathsf{S}_{n_2}(x)$, this case actually does not happen. First, for all these clusters, we should have $p = n_2$. Indeed, otherwise $\{\mathrm{k}_1, \dots, \mathrm{k}_p, \mathrm{k}\}$ would form a cluster around x containing $p + 1 \le n_2$ independent vectors. We would then

conclude as in case 2 that $|\mathrm{k}\cdot(\omega+\omega')| \le (4r^2+2)\delta$, so that, by Lemma 3, $\omega+\omega' \in \mathsf{B}(\mathrm{k}_1,\dots,\mathrm{k}_p,\mathrm{k})$ if L has been chosen large enough. This would contradict the assumption ensuring that we are in case 3. So $p = n_2$ should hold. Writing $\omega'' = \omega+\omega'$, we should then conclude from the definition of $\mathsf{B}(\mathrm{k}_1,\dots,\mathrm{k}_p)$ that, for $1 \le j \le n_2$,

$$|\mathrm{k}_j\cdot\omega''| \;=\; |\mathrm{k}_j|_2\big|\omega'' - P(\omega'',\pi(\mathrm{k}_j))\big|_2 \;\le\; |\mathrm{k}_j|_2\big|\omega'' - P\big(\omega'',\pi(\mathrm{k}_1)\cap\dots\cap\pi(\mathrm{k}_{n_2})\big)\big|_2 \;\le\; |\mathrm{k}_j|_2 L^{n_2}\delta$$

But then

$$|\mathrm{k}_j\cdot\omega| \;=\; |\mathrm{k}_j\cdot(\omega+\omega') - \mathrm{k}_j\cdot\omega'| \;\le\; |\mathrm{k}_j\cdot\omega''| + |\mathrm{k}_j\cdot\omega'| \;\le\; |\mathrm{k}_j|_2 L^{n_2}\delta + 4r^2\delta \;\le\; L^{n_2+1}\delta$$

if L is large enough. This would contradict $\omega \notin \mathsf{S}_{n_2}(x)$. $\square$

5 Proof of Theorem 1: the rotor chain

Let $a \in \mathbb{Z}_N$ be given by hypothesis. Let us assume that the dynamics is generated by Hamiltonian (1).

5.1 New decomposition of the Hamiltonian

The original decomposition of the Hamiltonian leading to the definition of the current $\epsilon J_{a,a+1}$ is given by

$$H \;=\; H^{\mathrm{O}}_{\le a} + H^{\mathrm{O}}_{>a} \;=\; \sum_{x\le a} H_x \;+\sum_{x>a} H_x. \tag{44}$$

We will now obtain a new decomposition of the Hamiltonian that is equivalent to the one above from the point of view of the conductivity, but leading to an instantaneous current that vanishes for most of the configurations in the Gibbs state at temperature T.

Let $n_3 \ge 1$. For $x \in \mathrm{B}(a,n_3)$, we define

$$\vartheta_{a,x} \;=\; \frac{1}{\mathcal{N}}\Big(\Big(\prod_{y\in\mathrm{B}(a,n_3)}\theta_y\Big)\,\delta_{a,x} \;+\; \Big(1-\prod_{y\in\mathrm{B}(a,n_3)}\theta_y\Big)\,\theta_x\Big), \tag{45}$$

$$\vartheta_{a,*} \;=\; \frac{1}{\mathcal{N}}\prod_{y\in\mathrm{B}(a,n_3)}(1-\theta_y). \tag{46}$$

with the normalization factor

$$\mathcal{N} \;=\; \Big(\prod_{y\in\mathrm{B}(a,n_3)}\theta_y\Big) + \Big(1-\prod_{y\in\mathrm{B}(a,n_3)}\theta_y\Big)\Big(\sum_{x\in\mathrm{B}(a,n_3)}\theta_x\Big) + \prod_{y\in\mathrm{B}(a,n_3)}(1-\theta_y) \tag{47}$$

chosen so that

$$\sum_{x\in\mathrm{B}(a,n_3)}\vartheta_{a,x} \;+\; \vartheta_{a,*} \;=\; 1,$$

and satisfying $\mathcal{N} \ge 1$. We then define

$$\widetilde{H}_{\le a} \;=\; \sum_{x\in\mathrm{B}(a,n_3)}\vartheta_{a,x}\sum_{y\le x}\widetilde{H}_y \;+\; \vartheta_{a,*}\sum_{y\le a}\widetilde{H}_y, \tag{48}$$

$$\widetilde{H}_{>a} \;=\; \sum_{x\in\mathrm{B}(a,n_3)}\vartheta_{a,x}\sum_{y>x}\widetilde{H}_y \;+\; \vartheta_{a,*}\sum_{y>a}\widetilde{H}_y. \tag{49}$$

It holds that

$$\widetilde{H} \;=\; \widetilde{H}_{\le a} + \widetilde{H}_{>a}.$$

By the first point of Proposition 1, we finally define a new decomposition

$$H \;=\; H_{\le a} + H_{>a} \;=\; \mathcal{T}_{n_1}(R\widetilde{H}_{\le a}) + \mathcal{T}_{n_1}(R\widetilde{H}_{>a}). \tag{50}$$

5.2 Definition of U_a and G_a

With the definitions (44) and (50), and applying the second point of Proposition 1, we find that

$$\epsilon J_{a,a+1} \;=\; L_H H^{\mathrm{O}}_{>a} \;=\; L_H(H^{\mathrm{O}}_{>a} - H_{>a}) + L_H H_{>a} \tag{51}$$

$$\;=\; L_H(H^{\mathrm{O}}_{>a} - H_{>a}) + \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a}) + \epsilon^{n_1+1} L_V \sum_{k=0}^{n_1} R^{(n_1-k)}\widetilde{H}^{(k)}_{>a}. \tag{52}$$

Let us call n_0 the number n appearing in the statement of the Theorem. We define

$$U_a \;=\; H^{\mathrm{O}}_{>a} - H_{>a} \;-\; \langle H^{\mathrm{O}}_{>a} - H_{>a}\rangle_T, \tag{53}$$

$$\epsilon^{n_0+1} G_a \;=\; \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a}) + \epsilon^{n_1+1} L_V \sum_{k=0}^{n_1} R^{(n_1-k)}\widetilde{H}^{(k)}_{>a}. \tag{54}$$

We notice that $\langle G_a\rangle_T = 0$ since $\epsilon^{n_0+1}\langle G_a\rangle_T = \langle L_H H_{>a}\rangle_T = 0$, by invariance of the Gibbs state.

5.3 Locality

Let us show that the functions U_a and G_a are local, meaning that they depend only on variables indexed by z with $|z-a| \le \mathrm{C}_{n_0}$, for some constant $\mathrm{C}_{n_0} < +\infty$. To study U_a we observe that

$$H^{\mathrm{O}}_{>a} - H_{>a} \;=\; -(H^{\mathrm{O}}_{\le a} - H_{\le a}).$$

Let us see that $H^{\mathrm{O}}_{>a} - H_{>a}$ depends only on variables indexed by z with $z \ge a - (n_3 + (n_2+5)r)$. The function $H^{\mathrm{O}}_{>a}$ only depends on variables indexed by z with $z \ge a$. To analyze $H_{>a}$ defined by (50), we first notice that the functions $\vartheta_{a,x}$, with $x \in \mathrm{B}(a,n_3)$, and $\vartheta_{a,*}$, defined by (45-46), only depend on variables indexed by z with $z \ge a - (n_3 + 4r + n_2 r)$. By (49), the same holds true for $\widetilde{H}_{>a}$, since, for any $x \in \mathbb{Z}_N$, the functions $\widetilde{H}_x$ only depend on variables indexed by z with $z \ge x - r$. By (24), we conclude that $R\widetilde{H}_{>a}$, and so $H_{>a}$, only depends on variables indexed by z with $z \ge a - (n_3 + 4r + n_2 r + r)$. The same holds thus for $H^{\mathrm{O}}_{>a} - H_{>a}$. We could similarly show that $H^{\mathrm{O}}_{\le a} - H_{\le a}$ only depends on variables indexed by z with $z \le a + (n_3 + (n_2+5)r)$. We conclude that U_a defined by (53) only depends on variables indexed by z with $|z-a| \le (n_3 + (n_2+5)r)$.

We then readily conclude that G_a is local as well, since, going back to (51), we see that $\epsilon^{n_0+1}G_a$ is the sum of two local functions:

$$\epsilon^{n_0+1} G_a \;=\; L_H H_{>a} \;=\; \epsilon J_{a,a+1} - L_H(H^{\mathrm{O}}_{>a} - H_{>a}).$$

5.4 An expression for $L_{\widetilde{H}}\widetilde{H}_{>a}$

We have

$$L_{\widetilde{H}}\widetilde{H}_{>a} = \sum_{x\in \mathrm{B}(a,n_3)} \vartheta_{a,x}\cdot\Big(L_{\widetilde{H}}\sum_{y>x}\widetilde{H}_y\Big) + \vartheta_{a,*}\cdot\Big(L_{\widetilde{H}}\sum_{y>a}\widetilde{H}_y\Big)$$
$$+\sum_{x\in \mathrm{B}(a,n_3)} (L_{\widetilde{H}}\vartheta_{a,x})\sum_{y>x}\widetilde{H}_y + (L_{\widetilde{H}}\vartheta_{a,*})\sum_{y>a}\widetilde{H}_y. \tag{55}$$

Let us show that the terms in the first sum in the right hand side vanish, i.e.

$$\vartheta_{a,x}\cdot\Big(L_{\widetilde{H}}\sum_{y>x}\widetilde{H}_y\Big) = 0 \quad \text{for all} \quad x\in\mathrm{B}(a,n_3). \tag{56}$$

Thanks to the presence of the operator $\mathcal{R}$ in (23), and thanks to (24), we decompose $\widetilde{H}=\sum_x\widetilde{H}_x$ where $\widetilde{H}_x$ takes the form

$$\widetilde{H}_x(q,\omega) = \sum_{\substack{\mathrm{k}\in K_r:\\ \mathrm{supp}\mathrm{k}\subset\mathrm{B}(x,r)}} \rho_\delta(\mathrm{k}\cdot\omega)\widetilde{V}_x(\mathrm{k},\omega)\mathrm{e}^{i\mathrm{k}\cdot q} = \widetilde{D}_x(\omega) + \sum_{\substack{\mathrm{k}\in K_r:\mathrm{k}\neq 0\\ \mathrm{supp}\mathrm{k}\subset\mathrm{B}(x,r)}} \rho_\delta(\mathrm{k}\cdot\omega)\widetilde{V}_x(\mathrm{k},\omega)\mathrm{e}^{i\mathrm{k}\cdot q}, \tag{57}$$

where we have singled out the $\mathrm{k}=0$ mode by setting $\widetilde{D}_x(\omega)=\widetilde{V}_x(0,\omega)$. We note that $\widetilde{D}_x$ depends only on variables indexed by z with $z\in\mathrm{B}(x,r)$.

Using the more handy notation $L_fg=\{f,g\}$, we compute misprint corrected

$$L_{\widetilde{H}}\sum_{y>x}\widetilde{H}_y = \Big\{\sum_{z\le x}\widetilde{H}_z,\sum_{y>x}\widetilde{H}_y\Big\} = \sum_{z\le x<y:|z-y|\le 2r}\Big\{\widetilde{H}_z,\widetilde{H}_y\Big\} \tag{58}$$

where we have used the fact that the Poisson bracket of two functions depending on variables indexed by points belonging to different, non-intersecting, subsets of $\mathbb{Z}_N$, vanishes. Relation (56) surely holds if ω is such that $\vartheta_{a,x}(\omega)=0$. Let us thus take ω such that $\vartheta_{a,x}(\omega)>0$, which implies $\theta_x(\omega)>0$. It then follows from the first point of Proposition 2 that all the factors $\rho_\delta(\mathrm{k}\cdot\omega)$ appearing when the operators $\widetilde{H}_z,\widetilde{H}_y$ in (58) are written as in (57), vanish. Therefore, when $\vartheta_{a,x}(\omega)>0$, the expression (58) equals misprint corrected

$$\sum_{z\le x<y:|z-y|\le 2r}\Big\{\widetilde{D}_z,\widetilde{D}_y\Big\}$$

However, since $\widetilde{D}_z,\widetilde{D}_y$ depend only on the ω-variables, their Poisson bracket vanishes. This establishes (56).

Thanks to (56), the Poisson bracket (55) is rewritten as

$$L_{\widetilde{H}}\widetilde{H}_{>a} = \vartheta_{a,*}\Big\{\sum_{a-r\le x\le a}\widetilde{H}_x,\sum_{a<y\le a+r}\widetilde{H}_y\Big\}$$
$$+\sum_{x\in\mathrm{B}(a,n_3)}(L_{\widetilde{H}}\vartheta_{a,x})\sum_{x<y\le a+n_3}\widetilde{H}_y + (L_{\widetilde{H}}\vartheta_{a,*})\sum_{a<y\le a+n_3}\widetilde{H}_y. \tag{59}$$

5.5 Definition of an exceptional set $\mathsf{Z}\subset\Omega$

Let $\mathsf{Z}\subset\Omega$ be such that $(\omega,q)\in\mathsf{Z}$ if and only if theres exists n_2 linearly independent vectors $\mathrm{k}_1,\dots,\mathrm{k}_{n_2}\in K_r$ such that $(\cup_j\mathrm{supp}(\mathrm{k}_j))\subset\mathrm{B}(a,2n_3)$ and such that $|\mathrm{k}_j\cdot\omega|\le L^{n_2+1}\delta$ for $1\le j\le n_2$. The set Z is closed.

Lemma 4. *If, given n_1 and n_2, the numbers L and n_3 have been taken large enough, then*

1. *If $L_{\widetilde{H}}\widetilde{H}_{>a}(\omega,q)\neq 0$, then $(\omega,q)\in\mathsf{Z}$.*

2. *There exists $\mathrm{C}=\mathrm{C}(L,n_1,n_2,n_3)<+\infty$ such that $\langle\chi_{\mathsf{Z}}\rangle_T\leq \mathrm{C}\delta^{n_2}$.*

Proof. Let us start with the first point. From (59), we conclude that, if $L_{\widetilde{H}}\widetilde{H}_{>a}(\omega,q)\neq 0$, then at least one of the following quantities needs to be non-zero: $\vartheta_{a,*}(\omega)$, or $L_{\widetilde{H}}\vartheta_{a,*}(\omega,q)$, or $L_{\widetilde{H}}\vartheta_{a,x}(\omega,q)$ for some $x\in\mathrm{B}(a,n_3)$. In fact, since $0\leq\prod_{x\in\mathrm{B}(a,n_3)}(1-\theta_x)\leq 1$, and since $L_{\widetilde{H}}$ is a differential operator, $L_{\widetilde{H}}\prod_{x\in\mathrm{B}(a,n_3)}(1-\theta_x)(\omega,q)\neq 0$ implies $\prod_{x\in\mathrm{B}(a,n_3)}(1-\theta_x)(\omega)\neq 0$. Therefore, by inspection of the definitions (45) and (46), the condition $L_{\widetilde{H}}\widetilde{H}_{>a}(\omega,q)\neq 0$ implies actually

$$\prod_{x\in\mathrm{B}(a,n_3)}(1-\theta_x)\neq 0 \qquad\text{or}\qquad L_{\widetilde{H}}\theta_x(\omega,q)\neq 0\quad\text{for some}\quad x\in\mathrm{B}(a,n_3).$$

If $\prod_{x\in\mathrm{B}(a,n_3)}(1-\theta_x)\neq 0$, then $\theta_x(\omega)<1$ for all $x\in\mathrm{B}(a,n_3)$. If $\theta_x(\omega)<1$, there exists then, by the definition (32), some $\omega'\in\mathbb{R}^N$ such that $\max_y|\omega'_y|\leq 2\delta$, and such that $\omega''=\omega+\omega'\in\mathsf{R}(x)$. There exists therefore a cluster $\{\mathrm{k}_1,\dots,\mathrm{k}_p\}$ around x, with $p\leq n_2$, such that (31) holds. This implies

$$|\mathrm{k}_1\cdot\omega''| \;=\; |\mathrm{k}_1|_2\big|\omega''-P(\omega'',\pi(\mathrm{k}_1))\big|_2 \;\leq\; |\mathrm{k}_1|_2\big|\omega''-P\big(\omega'',\pi(\mathrm{k}_1)\cap\dots\cap\pi(\mathrm{k}_p)\big)\big|_2 \;\leq\; |\mathrm{k}_1|_2L^p\delta$$

and therefore $|\omega\cdot\mathrm{k}_1|\leq|\mathrm{k}_1|_2L^p\delta+r^2\delta\leq L^{n_2+1}\delta$ if L is large enough and using that $p\leq n_2$. It holds by definition of a cluster around x that $\mathrm{supp}(\mathrm{k}_1)\subset\mathrm{B}(x,4r)$. Let us now take another x' such that $\theta_{x'}>0$ and $|x-x'|>4r$. Then the same reasoning gives a vector $\mathrm{k}_1'\neq\mathrm{k}_1$ satisfying again $|\omega\cdot\mathrm{k}_1'|\leq L^{n+1}\delta$. By taking n_3 large enough, we can find n_2 linearly independent vectors and thus guarantee that $\omega\in\mathsf{Z}$.

Suppose now that $L_{\widetilde{H}}\theta_x(\omega,q)\neq 0$ for some $x\in\mathrm{B}(a,n_3)$. It then follows from the second assertion of Proposition 2 that $\omega\in\mathsf{S}(x)$, so that, by definition, there exists a cluster $\{\mathrm{k}_1,\dots,\mathrm{k}_{n_2}\}$ around x such that $|\omega\cdot k_j|\leq L^{n_2+1}\delta$. This implies $\omega\in\mathsf{Z}$.

We now move to the second claim of the Lemma. Since the function χ_{Z} depends only on the variables ω_x with $x\in\mathrm{B}(a,2n_3)$, and since the Gibbs measure factorizes with respect to the variables ω_y $(y\in\mathbb{Z}_N)$, we get proof simplified

$$\langle\chi_{\mathsf{Z}}\rangle_T \;=\; \frac{\int\chi_{\mathsf{Z}}(\omega)\prod_{x\in\mathrm{B}(a,2n_3)}\mathrm{e}^{-\omega_x^2/T}\,\mathrm{d}\omega_x}{\int\prod_{x\in\mathrm{B}(a,2n_3)}\mathrm{e}^{-\omega_x^2/T}\,\mathrm{d}\omega_x} \;\leq\; \mathrm{C}(n_3)\int\chi_{\mathsf{Z}}(\omega)\prod_{x\in\mathrm{B}(a,2n_3)}\mathrm{e}^{-\omega_x^2/T}\,\mathrm{d}\omega_x.$$

The result follows by a straightforward computation that exploits that the set Z is determined by n_2 constraints. $\square$

5.6 Bounds on the norms of U_a and G_a

Let $\partial_\sharp$ be the derivative with respect to any ω_z or q_z with $z\in\mathbb{Z}_N$, or even no derivative at all $(\partial_\sharp f=f)$. To lighten some notations, we again use the shorthand $g\sim\delta^{-n}$ with the same meaning as in the proof of (25) and (26). ***Rem. by W:*** **I found it confusing to repeat this without clearly stating that it is a repetition.** ***EOR*** We now allow δ to depend on ϵ, and we fix the values of n_1 and n_2.

Let us first obtain the bounds $\langle U_a^2\rangle_T \le \mathrm{C}_{n_0}\epsilon^{1/4}, \langle(\partial_\sharp U_a)^2\rangle_T \le \mathrm{C}_{n_0}\epsilon^{-1/4}$ (recall that we denote by n_0 the number n appearing in Theorem 1). We use (25) and (26), and the fact that the functions $\vartheta_{a,x}, \vartheta_{a,*}$ are bounded, to see that the local function $H^{\mathrm{O}}_{>a} - H_{>a}$ takes the form First part rewritten

$$H^{\mathrm{O}}_{>a} - H_{>a} \;=\; \sum_{n=0}^{n_1}\epsilon^n (H^{\mathrm{O}}_{>a} - H_{>a})^{(n)} \tag{60}$$

$$= \; H^{\mathrm{O}}_{>a} - \mathcal{T}_{n_1}\big(R\widetilde{H}_{>a}\big) \;=\; H^{\mathrm{O}}_{>a} - \sum_{n=0}^{n_1}\epsilon^n\sum_{k=0}^{n} R^{(n-k)}\widetilde{H}^{(k)}_{>a} \;\sim\; \sum_{n=0}^{n_1}\epsilon^n \delta^{-2(n-1)}. \tag{61}$$

Let us fix $\delta = \epsilon^{1/4}$. Let us start with the term corresponding to $n=0$ in (60). From (61) we have

$$(H^{\mathrm{O}}_{>a} - H_{>a})^{(0)} = \sum_{x>a} D_x - \tilde{H}^{(0)}_{>a}$$
$$= \sum_{x>a} D_x - \sum_{x\in \mathrm{B}(a,n_3)} \vartheta_{a,x}\sum_{y>x} D_y - \vartheta_{a,*}\sum_{y>a} D_y$$

Let $\mathsf{W}\subset\Omega$ be the set containing all (ω,q) such that $\theta_x(\omega) < 1$ for some $x\in\mathrm{B}(a,n_3)$. By inspection of the definitions (45) and (46), we have

$$\vartheta_{a,x}(\omega) = \delta_{a,x}, \qquad \vartheta_{a,*}(\omega) = 0, \qquad (H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}(\omega,q) = 0, \qquad \text{for } (\omega,q)\in\Omega\setminus\mathsf{W}$$

Therefore $(H^{\mathrm{O}}_{>a} - H_{>a})^{(0)} = \chi_{\mathsf{W}}\cdot(H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}$. So, arguing as in the proof of Lemma 4, we find that, since $\chi_{\mathsf{W}}\cdot(H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}$ depends ony on the variables ω_x, with $|x-a|\le\mathrm{C}$,

$$\Big\langle\Big((H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}\Big)^2\Big\rangle_T \;\le\; \mathrm{C}\int\chi_{\mathsf{W}}(\omega)\,\Big((H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}\Big)^2(\omega)\prod_{x:|x-a|\le\mathrm{C}}\mathrm{e}^{-\omega_x^2/T}\,\mathrm{d}\omega_x$$
$$\le\; \mathrm{C}'\int\chi_{\mathsf{W}}(\omega)\prod_{x:|x-a|\le\mathrm{C}}\mathrm{e}^{-\omega_x^2/2T}\,\mathrm{d}\omega_x \;\le\; \mathrm{C}''\delta \;=\; \mathrm{C}''\epsilon^{1/4}, \tag{62}$$

where, to get the last inequality, we used that, for any $(\omega,q)\in\mathsf{W}$, there is at least one $\mathrm{k}\in K_r$, with $\mathrm{supp}(\mathrm{k})\subset\mathrm{B}(a,n_3)$, such that $|\omega\cdot\mathrm{k}|\le L^{n_2+1}\delta$. Similarly

$$\Big\langle\Big(\partial_\sharp(H^{\mathrm{O}}_{>a} - H_{>a})^{(0)}\Big)^2\Big\rangle_T \;\le\; \mathrm{C}\frac{\delta}{\delta^2} \;\le\; \mathrm{C}\epsilon^{-1/4}. \tag{63}$$

Next, we conclude from (17) and (61) that for $1\le n\le n_1$, we have

$$\Big\langle\Big(\epsilon^n\partial_\sharp(H^{\mathrm{O}}_{>a} - H_{>a})^{(n)}\Big)^2\Big\rangle_T \;\le\; \mathrm{C}\left(\epsilon^n\delta^{-(2n-1)}\right)^2 \;\le\; \mathrm{C}. \tag{64}$$

Using (62-64) in (60), we deduce

$$\langle(H^{\mathrm{O}}_{>a} - H_{>a})^2\rangle_T \;\le\; \mathrm{C}\epsilon^{1/4}, \qquad \langle(\partial_\sharp(H^{\mathrm{O}}_{>a} - H_{>a}))^2\rangle_T \;\le\; \mathrm{C}\epsilon^{-1/4},$$

from which the claimed bounds on U_a follow.

Next, to obtain the bound $\langle(\partial_\sharp G_a)^2\rangle_T\le\mathrm{C}_{n_0}$, we start from the definition (54) and we note that both terms in this definition are local; for $\mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})$ this follows from the explicit expression in Section 5.4 and the other term is then local as a difference of local terms. We compute factors 2 and $\partial_\sharp$ added

$$\langle(\partial_\sharp G_a)^2\rangle_T \;\le\; 2\epsilon^{-2(n_0+1)}\Big\langle\big(\partial_\sharp\mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})\big)^2\Big\rangle_T \;+\; 2\epsilon^{2(n_1-n_0)}\Big\langle\Big(\partial_\sharp L_V\sum_{k=0}^{n_1}R^{(n_1-k)}\widetilde{H}^{(k)}_{>a}\Big)^2\Big\rangle_T \tag{65}$$

We look first at the second term and conclude, by means of (25) and (26) that it is of the form estimate corrected

$$\partial_\sharp L_V \sum_{k=0}^{n_1} R^{(n_1-k)} \widetilde{H}^{(k)}_{>a} \sim \delta^{-2n_1}.$$

We thus obtain, using locality, corrected

$$\epsilon^{2(n_1-n_0)} \Big\langle \Big(\partial_\sharp L_V \sum_{k=0}^{n_1} R^{(n_1-k)} \widetilde{H}^{(k)}_{>a} \Big)^2 \Big\rangle_T \le \mathrm{C}_{n_0} \quad \text{if} \quad \delta = \epsilon^{1/4} \quad \text{and} \quad \cancel{n_1 = 3n_0 + 1} n_1 = 2n_0.$$

We then analyze the first term in (65). By the first point of Lemma 4, the function $L_{\widetilde{H}} \widetilde{H}_{>a}$ vanishes on the open set $\Omega \setminus \mathsf{Z}$, so that $\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})$ vanishes on this set as well. Using Cauchy-Schwarz inequality, and the second point of Lemma 4, we conclude that

$$\big\langle \big(\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})\big)^2 \big\rangle_T = \big\langle \big(\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})\big)^2 \chi_{\mathsf{Z}} \big\rangle_T \le \mathrm{C}\delta^{n_2/2} \big\langle \big(\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})\big)^4 \big\rangle_T^{1/2}.$$

Using (25, 26) and the fact that $\vartheta_{a,x}, \vartheta_{a,*}$ are bounded functions of ω/δ, we find that corrected

$$\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a}) = \partial_\sharp \sum_{n=0}^{n_1} \epsilon^n \sum_{m=0}^{n} R^{(n-m)} \sum_{j=0}^{m} \big\{ \widetilde{H}^{(m-j)}, \widetilde{H}^{(j)}_{>a} \big\} \sim \sum_{n=0}^{n_1} \epsilon^n \delta^{-2n+2} \sim \delta^2.$$

We conclude that $\delta = \epsilon^{1/4}$ guarantees that minor change

$$\big\langle \big(\partial_\sharp \mathcal{T}_{n_1}(RL_{\widetilde{H}}\widetilde{H}_{>a})\big)^2 \big\rangle_T \le \mathrm{C}\delta^{n_2/2+4} \le \mathrm{C}\epsilon^{n_2/8+1}.$$

taking $n_2 = 16n_0 + 8$, we conclude that $\langle (\partial_\sharp G_a)^2 \rangle_T \le \mathrm{C}_{n_0}$. This shows Theorem 1 in the case where H is given by (1). □

6 Proof of Theorem 1 for the NLS chain

The proof of the theorem given for rotors can readily be taken over to the non-linear Schrödinger chain by a adapting to this case the definition of the space $\mathcal{S}(\Omega)$ given in Section 3.

We simply view functions on $\Omega = \mathbb{C}^N$ as functions on $\mathbb{R}^{2N}$. We use the notation $\omega = (\omega_x)_{x \in \mathbb{Z}_N} = (|\psi_x|^2)_{x \in \mathbb{Z}_N}$, and we observe that now $\omega \in (\mathbb{R}_+)^N$. We will say that $f \in \mathcal{S}(\Omega)$ if the following conditions are met for some $r(f) < +\infty$: We first ask, as before, that f is smooth, that f and all its derivatives are of polynomial growth and that f can be expressed as a sum of local terms, each of which is indexed by z in a ball or radius $r(f)$. We next replace (16) by the requirement that f takes the form

$$f(\psi) = \sum_{\mathrm{k} \in \mathbb{Z}_N} \widehat{f}(\mathrm{k},\omega) \psi^{\mathrm{k}} \qquad \text{with} \qquad \widehat{f}(\mathrm{k},\omega) = 0 \quad \text{if} \quad \max_x |\mathrm{k}_x| \ge r(f),$$

where we have used the notation

$$\psi^{\mathrm{k}} = \prod_{x \in \mathbb{Z}_N} \widetilde{\psi}_x^{\mathrm{k}_x} \quad \text{with} \quad \widetilde{\psi}_x^{\mathrm{k}_x} = \begin{cases} \psi_x^{\mathrm{k}_x} & \text{if} \quad \mathrm{k}_x > 0, \\ 1 & \text{if} \quad \mathrm{k}_x = 0, \\ \overline{\psi}_x^{\mathrm{k}_x} & \text{if} \quad \mathrm{k}_x < 0. \end{cases}$$

Rem. by W: **misprints corrected** ***EOR*** It is then checked that the formulas involving explicitly the decomposition of f in its Fourier components is simply transposed by means of the substitution $\mathrm{e}^{i\mathrm{k}\cdot q} \mapsto \psi^{\mathrm{k}}$.

All the definitions and conclusions of Section 3 remain valid with obvious adjustments whereas the third claim of Proposition 1 can be omitted since it is only used to prove Theorem 3 stated for the rotor chain only.

Rem. by W: **Next passage has been added** ***EOR*** Let us comment on the 'obvious adjustments' needed in Section 3. The notation $g \sim \delta^{-n}$ is now understood to mean that g is a sum of terms of the form

$$\delta^{-n} b(\omega/\delta, \delta) f(\psi, \bar{\psi})$$

with b being exactly as in Section 3 and $f \in \mathcal{S}(\Omega)$. The definition of $\mathcal{R}$ and of the solution of the equation $L_D u = f$ is again simply taken over from Section 3. As before, it holds that if $g \sim \delta^{-n}, h \sim \delta^{-m}$, then

$$L_g h \sim \delta^{-m-n-1}.$$

Let us illustrate this. For simplicity consider $g = \delta^{-n} b_g(\omega/\delta), h = \delta^{-m} b_h(\omega/\delta)$, then

$$(L_g h)(\psi, \bar{\psi}) = -i\delta^{-n-m} \sum_x \frac{\partial}{\partial \bar{\psi}_x} b_g(\omega/\delta) \frac{\partial}{\partial \psi_x} b_h(\omega/\delta) + \dots (\psi \leftrightarrow \bar{\psi}) \tag{66}$$

$$= -i\delta^{-n-m} \sum_x \frac{\partial \omega_x}{\partial \bar{\psi}_x} \frac{\partial b_g(\omega/\delta)}{\partial \omega_x} \frac{\partial \omega_x}{\partial \psi_x} \frac{\partial b_h(\omega/\delta)}{\partial \omega_x} + \dots \tag{67}$$

$$= -i\delta^{-n-m-1} \sum_x \left\{ (\omega_x/\delta)\ b'_{g,x}(\omega/\delta) b'_{h,x}(\omega/\delta) - \dots \right\} \tag{68}$$

where $b'_{g,x}, b'_{h,x}$ are partial derivatives of b_g, b_h and the expression between $\{\dots\}$ is a smooth function of ω/δ.

The results of Section 4 can also be taken over without change. It is observed that the functions θ_x, $x \in \mathbb{Z}_N$, can still be defined for $\omega \in \mathbb{R}^N$, even though only their restriction to $(\mathbb{R}_+)^N$ is needed.

The arguments of Section 5 remain all valid as well, except for the claims relying on the fact that the Gibbs measure factorize with respect to the variables ω_x ($x \in \mathbb{Z}_N$). This includes the proof of the second claim of Lemma 4, the argument leading to (62,63), and the property that the polynomials that depend only on some fixed variables, are integrable with respect to the Gibbs measure, with bounds that do not depend on the volume. We now prove that $\langle \chi_{\mathsf{Z}} \rangle_T \leq \mathrm{C}\delta^{n_2}$ for the non-linear Schrödinger chain as well (the other properties are shown in the same way). Let us assume $T = 1$ for simplicity of notation. made more precise

We have

$$\langle \chi_{\mathsf{Z}} \rangle_1 \;=\; \frac{\int \chi_{\mathsf{Z}} \mathrm{e}^{-H(\psi)}\, \mathrm{d}\psi}{\int \mathrm{e}^{-H(\psi)}\, \mathrm{d}\psi}, \tag{69}$$

with $\mathrm{d}\psi = \mathrm{d}\Re(\psi)\mathrm{d}\Im(\psi)$. The function χ_{Z} only depends on the variables indexed by $x \in \mathrm{B}(a, 2n_3)$, and it is thus natural to factorize the integrals into three pieces. ***Rem. by W:*** **I thinkk in what follows you forgot the $1/2$ in front of the quartic term, all my changes are consequences of that** ***EOR***
For the numerator, we simply use the fact that $\mathrm{e}^{-\phi} \leq 1$ for all $\phi \geq 0$, to obtain that

$$\int \chi_{\mathsf{Z}} \mathrm{e}^{-H(\psi)}\, \mathrm{d}\psi \;\leq\; \int \prod_{x < a - 2n_3} \mathrm{e}^{-H'_x(\psi)}\, \mathrm{d}\psi_x \cdot \int \chi_{\mathsf{Z}} \prod_{x \in \mathrm{B}(a, 2n_3)} \mathrm{e}^{-H'_x(\psi)}\, \mathrm{d}\psi_x \cdot \int \prod_{x > a + 2n_3} \mathrm{e}^{-H'_x(\psi)}\, \mathrm{d}\psi_x,$$

where H'_x differs from H_x only for the boundary terms $x = a - 2n_3 - 1$ and $x = a + 2n_3$:

$$H'_{a-2n_3-1} \;=\; H_{a-2n_3-1} - \epsilon|\psi_{a-2n_3} - \psi_{a-2n_3-1}|^2 \quad \text{and} \quad H'_{a+2n_3} = H_{a+2n_3} - \epsilon|\psi_{a+2n_3+1} - \psi_{a+2n_3}|^2.$$

These definitions are chosen such that H'_{a-2n_3-1} does not depend on ψ_{a-2n_3} and H'_{a+2n_3} does not depend on ψ_{a+2n_3+1}. The middle integral is estimated as in the case of rotors:

$$\int \chi\mathrm{z} \prod_{x\in \mathrm{B}(a,2n_3)} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x \leq \mathrm{C}\int \chi\mathrm{z} \prod_{x\in \mathrm{B}(a,2n_3)} \mathrm{e}^{-\frac{1}{4}|\psi_x|^4}\,\mathrm{d}\psi_x \;\leq\; \mathrm{C}\,\delta^{n_2}.$$

for ϵ sufficiently small. We then similarly factorize the numerator in (69), using this time the bound $|\psi_x - \psi_{x+1}|^2 \leq 2(|\psi_x|^2 + |\psi_{x+1}|^2)$ for the boundary terms:

$$\int \mathrm{e}^{-H(\psi)}\,\mathrm{d}\psi \;\geq\; \int \prod_{x<a-2n_3} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x \cdot \int \prod_{x\in\mathrm{B}(a,2n_3)} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x \cdot \int \prod_{x>a+2n_3} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x$$

where H''_x differs from H_x only for the following boundary terms:

$$H''_{a-2n_3-1} \;=\; H_{a-2n_3-1} - \epsilon|\psi_{a-2n_3} - \psi_{a-2n_3-1}|^2 + 2\epsilon|\psi_{a-2n_3-1}|^2, \quad H''_{a-2n_3} \;=\; H_{a-2n_3} + 2\epsilon|\psi_{a-2n_3}|^2$$

as well as H''_{a+2n_3} and H''_{a+2n_3+1} that are defined similarly. The middle integral is bounded from below by a constant depending on n_3, using $\mathrm{e}^{-H_x} \geq \mathrm{e}^{-|\psi_x|^4}$ for ϵ sufficiently small.

To finish the proof, it is thus enough to show that there exists a constant $\mathrm{C} < +\infty$ such that

$$\frac{\int \prod_{x<a-2n_3} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x<a-2n_3} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x} \;\leq\; \mathrm{C} \qquad \text{and} \qquad \frac{\int \prod_{x>a+2n_3} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x>a+2n_3} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x} \;\leq\; \mathrm{C}.$$

Both cases are treated similarly, and we consider the second one only. Writing $b = a + 2n_3 + 1$, we have

$$\frac{\int \prod_{x\geq b} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x\geq b} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x} \;=\; \frac{\int \mathrm{e}^{2\epsilon|\psi_b|^2} \prod_{x\geq b} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x\geq b} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x}.$$

Brascamp-Lieb inequalities furnish a possible way to estimate this integral (see [7]). Since the function $z \mapsto |z|^4$ is not strictly convex at origin, we need however to slightly modify the measure at $x = b$. There exist constants $c, c' > 0$ small enough such that

$$\psi_b \;\mapsto\; \frac{1}{2}|\psi_b|^4 + c\Big(1 - \frac{1}{1+|\psi_b|^2}\Big) - c'|\psi_b|^2$$

is convex on $\mathbb{C}$. Letting then

$$H'''_b(\psi) \;=\; H''_b + c\Big(1 - \frac{1}{1+|\psi_b|^2}\Big)$$

and $H'''_x = H''_x$ for $x > b$, we have

$$\frac{\int \prod_{x\geq b} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x\geq b} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x} \;\leq\; \mathrm{C}\frac{\int \mathrm{e}^{2\epsilon|\psi_b|^2} \prod_{x\geq b} \mathrm{e}^{-H'''_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x\geq b} \mathrm{e}^{-H'''_x(\psi)}\,\mathrm{d}\psi_x}.$$

It is checked that

$$\psi \;\mapsto\; \sum_{x=b}^{(N-1)/2} H'''_x(\psi) - \Big(c'|\psi_b|^2 + \frac{\epsilon}{2}\sum_{x=b}^{(N-1)/2} |\psi_{x+1} - \psi_x|^2\Big)$$

is convex, with the convention that $\psi_{(N+1)/2} = \psi_{(N-1)/2}$. By Brascamp-Lieb inequality (Corollary 7 in [7]), followed by the change of variables $\phi_b = \psi_b$ and $\phi_{x+1} = \psi_{x+1} - \psi_x$ for $b \le x \le (N-3)/2$, we conclude that

$$\begin{aligned}
\frac{\int \prod_{x\ge b} \mathrm{e}^{-H'_x(\psi)}\,\mathrm{d}\psi_x}{\int \prod_{x\ge b} \mathrm{e}^{-H''_x(\psi)}\,\mathrm{d}\psi_x} \;&\le\; \mathrm{C}\,\frac{\int \mathrm{e}^{-(c'-2\epsilon)|\psi_b|^2}\prod_{x\ge b}\mathrm{e}^{-\frac{\epsilon}{2}|\psi_{x+1}-\psi_x|^2}\,\mathrm{d}\psi_x}{\int \mathrm{e}^{-c'|\psi_b|^2}\prod_{x\ge b}\mathrm{e}^{-\frac{\epsilon}{2}|\psi_{x+1}-\psi_x|^2}\,\mathrm{d}\psi_x} \qquad (\psi_{(N+1)/2} = \psi_{(N-1)/2})\\
&=\; \mathrm{C}\,\frac{\int \mathrm{e}^{-(c'-2\epsilon)|\phi_b|^2}\prod_{x\ge b}\mathrm{e}^{-\frac{\epsilon}{2}|\phi_x|^2}\,\mathrm{d}\phi_x}{\int \mathrm{e}^{-c'|\phi_b|^2}\prod_{x\ge b}\mathrm{e}^{-\frac{\epsilon}{2}|\phi_x|^2}\,\mathrm{d}\phi_x}\\
&=\; \mathrm{C}\,\frac{\int \mathrm{e}^{-(c'-2\epsilon)|\phi_b|^2}\mathrm{d}\phi_b}{\int \mathrm{e}^{-c'|\phi_b|^2}\mathrm{d}\phi_b} \;\le\; \mathrm{C}'
\end{aligned}$$

for some $\mathrm{C}' < +\infty$. $\square$

7 Proofs of Theorems 2, 3 and 4

The proof of Theorems 2 and 3 closely follows the proof of analog results in [14], itself inspired by [21]. We will need some decorrelation properties of the Gibbs measure. General results in [19] apply to the measures corresponding to the Hamiltonians (1) and (3), if ϵ is small enough for a given temperature T. Given $A, B \subset \mathbb{Z}_N$, let $d(A,B) = \min\{|x-y| : x \in A, y \in B\}$. Given a local function f on Ω, let $S(f) \subset \mathbb{Z}_N$ be the set of points such that f does only depend on variables indexed by points in $S(f)$. There exist constants $\mathrm{C} < +\infty$ and $c > 0$ such that given two smooth functions f and g on Ω satisfying $\langle f\rangle_T = \langle g\rangle_T = 0$, it holds that

$$|\langle fg\rangle_T| \;\le\; \mathrm{C}\mathrm{e}^{-cd(S(f),S(g))}\langle|\nabla f|^2\rangle_T^{1/2}\langle|\nabla g|^2\rangle_T^{1/2} \tag{70}$$

where $|\nabla f|^2 = \sum_{x\in\mathbb{Z}_N}\big(|\partial_{\omega_x} f|^2 + |\partial_{q_x} f|^2\big)$ for the rotor chain and $|\nabla f|^2 = \sum_{x\in\mathbb{Z}_N}\big(|\partial_{\psi_x} f|^2 + |\partial_{\bar\psi_x} f|^2\big)$ for the NLS chain. Strictly speaking, (70) is stated in [19] only in the case where the one-site phase space is $\mathbb{R}$, but the proof goes through without any changes in our case as well (our one-site phase space is $\mathbb{T}\times\mathbb{R}$, and $\mathbb{C}$ respectively) since the only genuine requirement is a Poincaré inequality for the one-site measure.

Proof of Theorem 2. Applying Theorem 1, we write

$$\begin{aligned}
\epsilon\int_0^{\epsilon^{-n}t}\mathfrak{J}_N(X^s_\epsilon)\,\mathrm{d}s \;&=\; \frac{\epsilon}{\sqrt N}\sum_{a\in\mathbb{Z}_N}\int_0^{\epsilon^{-n}t} J_{a,a+1}(X^s_\epsilon)\,\mathrm{d}s\\
&=\; \frac{1}{\sqrt N}\sum_{a\in\mathbb{Z}_N}\big(U_a(X^{\epsilon^{-n}t}_\epsilon) - U_a(X^0_\epsilon)\big) + \frac{\epsilon^{n+1}}{\sqrt N}\sum_{a\in\mathbb{Z}_N}\int_0^{\epsilon^{-n}t} G_a(X^s_\epsilon)\,\mathrm{d}s.
\end{aligned}$$

Therefore factors 2

$$\begin{aligned}
\Big\langle\Big(\epsilon\int_0^{\epsilon^{-n}t}\mathfrak{J}_N(X^s_\epsilon)\,\mathrm{d}s\Big)^2\Big\rangle_T \;&\le\; 2\Big\langle\Big(\frac{1}{\sqrt N}\sum_{a\in\mathbb{Z}_N}\big(U_a(X^{\epsilon^{-n}t}_\epsilon) - U_a(X^0_\epsilon)\big)\Big)^2\Big\rangle_T\\
&\quad+\; 2\Big\langle\Big(\frac{\epsilon^{n+1}}{\sqrt N}\sum_{a\in\mathbb{Z}_N}\int_0^{\epsilon^{-n}t} G_a(X^s_\epsilon)\,\mathrm{d}s\Big)^2\Big\rangle_T
\end{aligned}$$

We conclude by stationarity of the Gibbs measure, by the decorrelation inequality (70), and by the bounds (8) that $4 \to 2$

$$\Big\langle\Big(\frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}\big(U_a(X_\epsilon^{\epsilon^{-n}t})-U_a(X_\epsilon^0)\big)\Big)^2\Big\rangle_T \;\leq\; 2\Big\langle\Big(\frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}U_a\Big)^2\Big\rangle_T \;\leq\; \mathrm{C}\epsilon^{-1/2}.$$

Next, by Jensen's inequality, by the invariance of the Gibbs measure, by the decorrelation inequality (70), and by the bounds (8), we have

$$\begin{aligned}\Big\langle\Big(\frac{\epsilon^{n+1}}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}\int_0^{\epsilon^{-n}t}G_a(X_\epsilon^s)\,\mathrm{d}s\Big)^2\Big\rangle_T \;&\leq\; \epsilon^{2(n+1)}\epsilon^{-n}t\int_0^{\epsilon^{-n}t}\Big\langle\Big(\frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}G_a(X_\epsilon^s)\Big)^2\Big\rangle_T\mathrm{d}s\\ &=\; \epsilon^{2(n+1)}\epsilon^{-2n}t^2\Big\langle\Big(\frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}G_a\Big)^2\Big\rangle_T \;\leq\; \mathrm{C}\epsilon^2t^2.\end{aligned}$$

Finally we obtain that

$$\epsilon^{-m}\Big\langle\Big(\frac{\epsilon}{\sqrt{\epsilon^{-n}t}}\int_0^{\epsilon^{-n}t}\mathcal{J}_N(X_\epsilon^s)\,\mathrm{d}s\Big)^2\Big\rangle_T \;=\; \frac{\mathrm{C}\epsilon^{n-m}}{t}(\epsilon^{-1/2}+\epsilon^2t^2).$$

Because $n-m>0$, this quantity goes to zero when taking successively the limits $N\to\infty$, $\epsilon\to 0$ and $t\to\infty$. □

Proof of Theorem 3. Let us write $\mathsf{E}_T(\cdot)$ for $\langle\mathsf{E}(\cdot)\rangle_T$. Theorem 1 implies that, for any $a\in\mathbb{Z}_N$,

$$\epsilon J_{a,a+1} \;=\; L_HU_a+\epsilon^{n+1}SU_a+\epsilon^{n+1}G_a-\epsilon^{n+1}SU_a \;=\; \mathcal{L}U_a+\epsilon^{n+1}G_a-\epsilon^{n+1}SU_a$$

where $\mathcal{L}$ is defined by (9). Since G_a is local and antisymmetric under the exchange $\omega\mapsto-\omega$, there exists a local function F_a that solves the Poisson equation $SF_a=G_a$ and inherits of the properties of G_a (see Lemma 2 in [14]). To simplify notations, let us write

$$\mathcal{U}_N \;=\; \frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}U_a,\qquad \mathcal{K}_N \;=\; \frac{1}{\sqrt{N}}\sum_{a\in\mathbb{Z}_N}(F_A-U_a).$$

We find that factors 2

$$\mathsf{E}_T\Big(\frac{\epsilon}{\sqrt{t}}\int_0^t\mathcal{J}_N(\mathcal{X}_\epsilon^s)\,\mathrm{d}s\Big)^2 \;\leq\; 2\mathsf{E}_T\Big(\frac{1}{\sqrt{t}}\int_0^t\mathcal{L}\mathcal{U}_N(\mathcal{X}_\epsilon^s)\,\mathrm{d}s\Big)^2 \;+\; 2\epsilon^{2(n+1)}\mathsf{E}_T\Big(\frac{1}{\sqrt{t}}\int_0^tS\mathcal{K}_N(\mathcal{X}_\epsilon^s)\,\mathrm{d}s\Big)^2.$$

The first term is written as a sum of the variance of a stationary martingale and a rest term: factor 2

$$\begin{aligned}\mathsf{E}_T\Big(\frac{1}{\sqrt{t}}\int_0^t\mathcal{L}\mathcal{U}_a\Big)^2 \;&=\; \langle\mathcal{U}_N\cdot(-\epsilon^{n+1}S)\mathcal{U}_N)\rangle_T \;+\; \frac{1}{t}E_T\big(\mathcal{U}_N(\mathcal{X}_\epsilon^t)-\mathcal{U}_N(\mathcal{X}_\epsilon^0)\big)^2\\ &\leq\; \epsilon^{n+1}\big\langle\mathcal{U}_N\cdot(-S)\mathcal{U}_N)\big\rangle_T \;+\; \frac{2\langle\mathcal{U}_N^2\rangle_T}{t} \;\leq\; \mathrm{C}\epsilon^{-1/2}(\epsilon^{n+1}+1/t),\end{aligned}\tag{71}$$

where the last bound has been obtained as in the proof of Theorem 2. The second term involves a function $S\mathcal{K}$, that obviously lies in the image of the symmetric part S of the generator $\mathcal{L}$. A classical bound [18] yields

$$\begin{aligned}\epsilon^{2(n+1)}\mathsf{E}_T\Big(\frac{1}{\sqrt{t}}\int_0^tS\mathcal{K}_N(\mathcal{X}_\epsilon^s)\,\mathrm{d}s\Big)^2 \;&\leq\; \mathrm{C}\epsilon^{2(n+1)}\big\langle S\mathcal{K}_N\cdot(-\epsilon^{n+1}S)^{-1}S\mathcal{K}_N\big\rangle_T\\ &=\; \mathrm{C}\epsilon^{n+1}\big\langle S\mathcal{K}_N\cdot\mathcal{K}_N\big\rangle_T \;\leq\; \mathrm{C}'\epsilon^{n+1/2},\end{aligned}\tag{72}$$

where the last bound has been obtained as in the proof of Theorem 2. The theorem is obtained by taking the limit $N\to\infty$ and then the limit $t\to\infty$ in (71) and (72) □

Proof of Theorem 4. By the definition of the currents $J_{a,a+1}$ we get

$$L_H(H_I) \;=\; \epsilon J_{a_1,a_1+1} - \epsilon J_{a_2,a_2+1}$$

so that, by integrating over time the statement of Theorem 1,

$$H_I(X^t_\epsilon) - H_I \;=\; \sum_{j=1,2} (-1)^{j+1}\Big(U_{a_j}(X^t_\epsilon) - U_{a_j} + \epsilon^{n+1}\int_0^t \mathrm{d}s\, G_{a_j}(X^s_\epsilon)\Big).$$

By invariance of the Gibbs state, we have $\langle (U_{a_j}(X^t_\epsilon))^2\rangle_T \;=\; \langle U^2_{a_j}\rangle_T$ for $j\;=\;1,2$. Hence by analogous manipulations as those in the proof of Theorem 2, we get

$$\langle (H_I(X^t_\epsilon) - H_I)^2\rangle_T \;\leq\; \mathrm{C}(n)\sum_{j=1,2}\Big(\langle U^2_{a_j}\rangle_T \;+\; \epsilon^{2n+1}t^2\langle G^2_{a_j}\rangle_T\Big)\,.$$

The theorem now follows by the bounds on U_a, G_a stated in Theorem 1, upon taking $t = \epsilon^{-n}$. □

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